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RADAR SIGNATURE ANALYSIS OF EXTENDED TARGETS

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PREFACE

This Memorandum was prepared as part of RAND's continuing study of advanced radar signal processing methods. It extends previous work on optimum signal processing methods for determining the characteristics of a radar target by analyzing the target's radar signature. The results of the study are applicable to the analysis of earth satellites and, suitably extended, could be applied to radar astronomy and radar reconnaissance.

SUMMARY

A problem of considerable interest in a variety of applications is that of estimating the characteristics of a radar target by analyzing its radar signature. Applications of interest include satellite identification, radar astronomy, and radar ground mapping. Target characteristics of interest include not just range, range rate, and range acceleration, but also dimensions, shape, and rate and axis of rotation.

Although many techniques have been developed to attack these problems, the theory of optimum signal processing methods and of the optimum estimation accuracies to be expected, as a function of the noise level, the data set, and the a priori knowledge of the target parameters, is still not well developed.

This Memorandum extends the inventory of techniques for defining optimum signal processing methods and predicting the accuracies achieved. Chief attention is given to estimation of linear or quadratic functionals of the scatterer amplitudes (such as the average scattering cross section attributable to specific regions on the target); however, the estimation of other target parameters is also treated. The analysis is carried farthest for the signature analysis of rotating targets.

CONTENTS

I. INTRODUCTION

The analysis reported in this Memorandum was initiated to develop a theory which would predict the accuracy with which the characteristics of an extended radar target can be determined by analyzing its radar signature. "Extended" means extended compared to a wavelength, not necessarily compared to a radar resolution cell. In fact, some of the interesting cases are those in which the target dimensions are much smaller than a spatial resolution cell of the radar. Of primary interest are rotating targets, although consideration will not be entirely restricted to this case. Characteristics of interest to determine might include size, shape, rate of rotation, axis of rotation, or radar cross section of specified portions of the body. This problem is of interest in identification of earth satellites and in radar astronomy, where the target is extended but isolated; and also in ground mapping, where the target is an extended continuum. (In ground mapping, and in signature analysis of stabilized satellites, the targets are not, strictly speaking, rotating. However, even in these cases, the relative motion of the target with respect to the radar can often be considered, to a good approximation, a rotation about the line of sight, after mean translatory motion has been corrected for.) Another type of problem which can be considered within this framework is the estimation, in the presence of noise, of the illumination pattern of a rotating antenna by observations of its far field.

A great deal of literature exists on this subject. Many special methods have been evolved, and a certain amount of theory has been

developed to predict the effects of various relevant factors on the success which can be obtained. Nevertheless, it cannot be said that a satisfactory theory has been developed. By a satisfactory theory, is meant one which will specify good or optimum signal processing methods to determine target characteristics, and will also predict the accuracy or reliability with which target characteristics of interest can be estimated, as a function of

a. types and statistical characteristics of errors,

b. characteristics of the data set, and

c. degree of a priori knowledge of the target.

Types of errors might include, for example, additive noise errors, phase errors, multiplicative amplitude noise, nonlinear distortions, etc. Characteristics of the data set would include waveform design, types(s) of radar (monostatic, multistatic, interferometric), extent of frequency and polarization diversity, scattering matrix elements to be measured, degree of coherence (not only temporally, at one sensor location, but from sensor to sensor), and length of observation interval or intervals. A priori knowledge about the target might take the form, to cite just a few examples, of knowledge that it is (from the electromagnetic viewpoint) a spherical surface, as in radar astronomy; that it is a stabilized body; or that it is a long, slender body.

At present, there are many relatively simple questions, in relatively simple contexts, which cannot be satisfactorily answered by existing theory. An example closely related to the situations which this Memorandum analyzes is the "low resolution" signature

analysis of radar targets such as earth satellites. It is well known that various characteristics of a target of this type, such as dimensions and to some extent shape, can be estimated rather well if the target can be viewed over a reasonable set of angular aspects, preferably but not necessarily including the broadside aspect; and that such estimation can be performed even if the spatial radar resolution cell is much larger than the physical dimensions of the target. Moreover, this does not necessarily require phase-coherent methods. However, there is at present no theory which will predict how the ability to estimate specified target characteristics depends on the noise level and on the set of aspects over which the body is observed. The effect of these factors must be determined empirically, by submitting signature records to analysts.

A satisfactory theory must also have the property that answers may be obtained in reasonable time and at reasonable cost on digital computers.

This Memorandum by no means succeeds in establishing a fully satisfactory general theory of target identification by radar signature analysis. In fact, a look at the problem rather quickly reveals that it is difficult to identify problems which are based on reasonable models of situations of practical interest and which, at the same time, can be solved, even using digital computers. The objective of this study is to contribute to the methodology available for attacking these problems by considering examples based on models reasonably near to physical reality--at least in the sense that the results obtained would give considerable insight into realistic situations--

and for which it is feasible to carry out the required calculations. Thus, what is derived here is not a general theory, but it is an addition to the arsenal of special theories for this class of problems. The actual implementation of the indicated calculations requires (except in very special cases) computer programs for solving certain integral equations; this has not been done as yet, although it will be clear that the necessary computations are well within the realm of feasibility.

POSSIBLE APPROACHES

Theoretical approaches to the class of problems just stated include the following:

- Modeling the electromagnetic signal scattered from the extended target as a noise process, and estimating the parameters of the noise statistics (such as parameters of the covariance function). Reference 1 provides good examples and a reasonably comprehensive bibliography. Essentially, this amounts to a model of the target as a reasonably numerous collection of independent scatterers.
- Modeling the scattered signal as depending predominantly on a finite set of parameters which can be identified with characteristics of the target motion or configuration, and applying parameter estimation theory, usually based on maximum-likelihood methods and Cramer-Rao bounds (equivalently, "information matrices") for estimating attainable accuracies.[2,3] For example, the target might be considered to consist of some reasonably small number of rigidly

connected point scatterers; the unknown parameters would then include the radar cross sections and relative positions of these scatterers.

Although many useful results have been obtained by these approaches, both have drawbacks when applied to the problem at hand. While the treatment of the signal as a noise process does not absolutely require the assumption that the received signal process is Gaussian or stationary, those treatments which attempt to define optimum signal processing methods, and to determine lower bounds on the estimation errors, usually assume that the noise is Gaussian, and often assume stationarity or quasi-stationarity in some sense.

However, many examples of the class of problems under consideration are characterized by the fact that the signal statistics are not Gaussian, or, even if they can reasonably be modeled as Gaussian, they are highly nonstationary. Moreover, this non-Gaussian or non-stationary behavior often involves essential aspects of the problem. For example, the signal may have a significant or dominating component arising from a very small number of scatterers. Or, more important for the examples to which this Memorandum is mainly devoted, the scatterer may act essentially as a line source or a plane source, i.e., there may be a small set of aspects (perhaps only one or two) at which all scatterers are in phase, producing what is called a "specular flash"; this specular flash is of great importance in the problem of estimating target characteristics.

The parameter estimation approach, insofar as it is based on Cramer-Rao bounds, information matrices, and maximum likelihood

methods, has the drawback that it is often difficult, in cases of nonlinear dependence on the parameters to be estimated, to say whether the accuracy bounds thus obtained are informative; that is, it is difficult to determine whether the Cramer-Rao lower bounds on error variance are reasonably close to the actual minimum attainable variances. In regular estimation cases, the Cramer-Rao variance bounds are asymptotically the minimum variances; "asymptotically" means for sufficiently large signal-to-noise ratio (SNR). However, as pointed out in Ref. 3, there are many pitfalls in determining how large the SNR must be for the Cramer-Rao bounds to be reasonably close to the minimum variances even if the estimation problem is "regular." It is definitely not true that output SNR which is sufficient for reliable target detection is sufficiently large to be in the asymptotic region just referred to. In fact, there are relatively simple, practical, and regular parameter estimation problems (a small, highly non-exhaustive set of examples is given in Ref. 3) in which the Cramer-Rao bound may be highly uninformative for SNRs which are adequate for detection purposes (i.e., the bound may be a small fraction of minimum attainable variance). This difficulty is even worse in more complex estimation problems, such as those involving the estimation of configuration parameters of connected sets of point scatterers.

The Bayes or Barankin approaches to parameter estimation avoid these pitfalls in theory, but ordinarily the computational difficulties associated with them are formidable. Nevertheless, the approach selected here is a Bayes approach. This approach is based on the minimum mean square error criterion. Theoretically, results can be

obtained for any signal-to-noise ratio. This is also practical in a number of cases of interest. In the interest of computational feasibility, however, it will in other cases be necessary to reintroduce some form of strong-signal assumption. The Bayes approach also has the advantage, over that of maximum likelihood, that it is possible to deal directly with functions of the unknown parameters; usually, functions of the parameters, rather than individual parameters, are of greatest interest.

Since a priori statistics must be associated with the unknown parameters in a Bayes approach, it is also possible to say that the scattered signal is considered to be a random process. Although most of the results will be derived under assumptions leading to a priori Gaussian statistics of the signal, some of the results can be extended to cases where the a priori signal statistics are mixtures of Gaussian statistics, i.e., they result from choosing a Gaussian probability measure from a specified set of such measures, with a probability distribution over the set of possible Gaussian measures. Also, it is possible to deal with cases in which there are peculiar forms of non-stationarity, e.g., those in which specular flashes occur. In addition, the models to be analyzed also are capable of reflecting various degrees of a priori knowledge of target characteristics.

Section II contains the derivation of the general methods to be employed. The target models under consideration in that section are not confined to rotating targets. Section III states, for a selection of rotating target models, the form taken by the more general expressions of Section II.

II. GENERAL RESULTS

A. INITIAL FORMULATION AND RESULTS

In the present section, the class of problems to be considered will not be restricted to rotating targets. Initially, it is supposed that the received signal, in the presence of additive noise, can be represented by

$$S(t) = \sum_i \alpha(\underset{\sim}{x}^{(i)}) F(t, \underset{\sim}{x}^{(i)}) + \epsilon(t) \tag{1}$$

where $\epsilon(t)$ is additive noise. $S(t)$ is observed over a known time interval or intervals.

Here, α is a function of the argument $\underset{\sim}{x}$, which in turn is a possibly multicomponent vector, ranging over a finite set of known vectors $\{\underset{\sim}{x}^{(i)}\}$, $i = 1, \ldots, n$. This could be regarded as an approximation to the representation of $S(t)$ as an integral, although such an interpretation is not always necessary. If such an interpretation is adopted, the required limiting procedures must be handled with great care (see Section II.C).

The parameter t represents time, although here, too, it is possible to consider t to be a multicomponent vector (although this will not be reflected in the notation). For example, if there are K different sensors, it would be possible to have $t = (t_1, \ldots, t_K)$ with t_k ranging over the observation times of the k^{th} sensor. Similarly, if a given sensor observes several different cross-polarizations, this could be formulated by considering t to be multidimensional. The vector $\underset{\sim}{x}$ will represent primarily position coordinates with respect to a coordinate system fixed to the target.

Other components may be adjoined to $\underset{\sim}{x}$ to represent internal velocity components of the target, if any, and also to differentiate between two phase components of the signal scattered from any given point on the body. This will all be illustrated by concrete examples in the next section.

The set of unknown parameters is $\alpha(\underset{\sim}{x}^{(i)})$, $i = 1, 2, \ldots, n$. The problem which will primarily be addressed is the estimation of real-valued functionals of $\underset{\sim}{\alpha}$, and most important, of linear and quadratic functionals of $\underset{\sim}{\alpha}$.

It is assumed that the a priori statistics of both $\{\alpha(\underset{\sim}{x})\}$ and $\{\varepsilon(t)\}$ are Gaussian, and that $\{\alpha(\underset{\sim}{x})\}$ is statistically independent of $\{\varepsilon(t)\}$. The assumption of Gaussian a priori statistics makes $S(t, \underset{\sim}{\alpha})$ a Gaussian process; later, cases will be considered in which F depends on additional unknown parameters $\underset{\sim}{\lambda}$, in which cases $\underset{\sim}{S}$ would no longer be a Gaussian process.*

It will further be assumed, for expository purposes, that $\{\alpha(\underset{\sim}{x})\}$ and $\{\varepsilon(t)\}$ have zero ensemble means, and that

$$E\left[\alpha(\underset{\sim}{x}^{(i)})\ \alpha(\underset{\sim}{x}^{(j)})\right] = A_{ij} \tag{2}$$

$$E\left[\varepsilon(t)\ \varepsilon(t')\right] = \varphi(t, t') \tag{3}$$

$$E\left[\alpha(\underset{\sim}{x}^{(i)})\ \varepsilon(t)\right] = 0 \tag{4}$$

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*The assumptions about the statistics of $\{\alpha(\underset{\sim}{x})\}$ are assumptions about the a priori state of knowledge used in forming Bayes estimates, and do not necessarily apply to any particular scattering target.

It is then possible to state the conditional (a posteriori) probability density function (p.d.f.) of $\underset{\sim}{\alpha}$ given $\underset{\sim}{S}$:

$$p(\underset{\sim}{\alpha} \mid \underset{\sim}{S}) = (2\pi)^{-\frac{n}{2}} \left[\det \underset{\sim}{B}\right]^{\frac{1}{2}} \tag{5}$$

$$\times \exp\left\{-\tfrac{1}{2} \sum_{i,j} B_{ij} \left[\alpha(\underset{\sim}{x}^{(i)}) - \hat{\alpha}(\underset{\sim}{x}^{(i)})\right] \left[\alpha(\underset{\sim}{x}^{(j)}) - \hat{\alpha}(\underset{\sim}{x}^{(j)})\right]\right\}$$

where

$$\hat{\alpha}(\underset{\sim}{x}^{(i)}) = \lim \sum_{\mu,\nu} \sum_{j} B^{-1}_{ij} F(t_\mu, \underset{\sim}{x}^{(j)}) S(t_\nu) \tag{6}$$

$$B_{ij} = A^{-1}_{ij} + \lim \sum_{\mu,\nu} \eta_{\mu\nu} F(t_\mu, \underset{\sim}{x}^{(i)}) F(t_\nu, \underset{\sim}{x}^{(j)}) \tag{7}$$

$$\eta_{\mu\nu} = (\mu, \nu)^{\text{th}} \text{ element of matrix inverse to } \varphi(t_\mu, t_\nu) \tag{8}$$

In these formulas, lim implies the limit as the points $\{t_\mu\}$ become dense in the observation interval. Such limits are rigorously defined if $\{\varepsilon(t)\}$ is continuous in the mean and the functions $F(t, \underset{\sim}{x}^{(i)})$ satisfy certain regularity conditions.[3] In Eqs. (6) and (7), B^{-1}_{ij} and A^{-1}_{ij} refer to the (i, j) elements of the matrix inverses of $\underset{\sim}{B}$ and $\underset{\sim}{A}$. There is a large body of literature on methods of evaluating expressions of the type appearing in Eqs. (6) and (7); Ref. 3 contains a partial summary of such methods.

It will prove convenient to write Eqs. (6) and (7), and similar equations below, in vector-matrix notation:

$$\underset{\sim}{\hat{\alpha}} = \underset{\sim}{B}^{-1} \underset{\sim}{F}^{T} \underset{\sim}{\eta} \underset{\sim}{S} \tag{9}$$

$$\underset{\sim}{B} = \underset{\sim}{A}^{-1} + \underset{\sim}{F}^{T} \underset{\sim}{\eta} \underset{\sim}{F} \tag{10}$$

Here, $\underset{\sim}{\hat{\alpha}}$ is an n-component vector with elements $\hat{\alpha}(\underset{\sim}{x}^{(i)})$; $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are $n \times n$ matrices; and

$$(\underset{\sim}{F}^T \underset{\sim}{\eta} \underset{\sim}{F})_{ij} = \lim \sum_{\mu,\nu} \eta_{\mu\nu} F(t_\mu, \underset{\sim}{x}^{(i)}) F(t_\nu, \underset{\sim}{x}^{(j)}) \tag{11}$$

$$= \int\int \eta(t, t') F(t, \underset{\sim}{x}^{(i)}) F(t', \underset{\sim}{x}^{(j)}) \, dt \, dt'$$

$$(\underset{\sim}{F}^T \underset{\sim}{\eta} \underset{\sim}{S})_i = \lim \sum_{\mu,\nu} \eta_{\mu\nu} F(t_\mu, \underset{\sim}{x}^{(i)}) S(t_\nu) \tag{12}$$

$$= \int\int \eta(t, t') F(t, \underset{\sim}{x}^{(i)}) S(t') \, dt \, dt'$$

The superscript T denotes a vector, matrix, or operator transpose. $\eta(t, t')$ is the operator inverse of $\varphi(t, t')$, here written symbolically as an integral kernel.

Now let β be any real-valued functional of $\underset{\sim}{\alpha}$ having a finite second moment with respect to the a priori p.d.f. of $\underset{\sim}{\alpha}$. Then $\hat{\beta}$, the Bayes optimum estimate of β, is the expected value of β with respect to $p(\underset{\sim}{\alpha} \mid \underset{\sim}{S})$. In particular, let linear and quadratic functionals be defined:

$$\beta_L = \sum_i \alpha(\underset{\sim}{x}^{(i)}) L(\underset{\sim}{x}^{(i)}) = \underset{\sim}{L} \cdot \underset{\sim}{\alpha} \tag{13}$$

$$\beta_Q = \sum_{i,j} \alpha(\underset{\sim}{x}^{(i)}) \alpha(\underset{\sim}{x}^{(j)}) Q(\underset{\sim}{x}^{(i)}, \underset{\sim}{x}^{(j)}) = \underset{\sim}{\alpha}^T \underset{\sim}{Q} \underset{\sim}{\alpha} \tag{14}$$

Then,

$$\hat{\beta}_L = \sum_i L(\underset{\sim}{x}^{(i)})\ \hat{\alpha}(\underset{\sim}{x}^{(i)}) = \underset{\sim}{L} \cdot \hat{\underset{\sim}{\alpha}} \tag{15}$$

$$\hat{\beta}_Q = \text{Trace}\left[\underset{\sim}{B}^{-1}\ \underset{\sim}{Q}\right] + \hat{\underset{\sim}{\alpha}}^T\ \underset{\sim}{Q}\ \hat{\underset{\sim}{\alpha}} \tag{16}$$

These formulas are obtained simply by taking the expected values of β_L or β_Q with respect to $p(\underset{\sim}{\alpha} \mid \underset{\sim}{S})$. Note that $\hat{\beta}_Q$ is not formed only by inserting the estimates $\hat{\alpha}(\underset{\sim}{x}^{(i)})$ into the expression for β_Q; an additional term is necessary. In a similar manner the Bayes estimates can easily be written down for third, fourth, or higher order forms in $\underset{\sim}{\alpha}$, applying well-known formulas for higher moments of Gaussian distributions in terms of their means and covariances. These formulas would be applied to find expected values with respect to $p(\underset{\sim}{\alpha} \mid \underset{\sim}{S})$; the resulting estimates would be functionals of $\hat{\underset{\sim}{\alpha}}$ and $\underset{\sim}{B}^{-1}$. These constitute a class of nonlinear estimation cases in which solutions valid for all signal-to-noise ratios are available.

B. DEPENDENCE ON ADDITIONAL PARAMETERS

It is of interest to see what happens when the signal statistics depend nonlinearly on an additional unknown parameter vector $\underset{\sim}{\lambda}$. This may happen if $F(t, \underset{\sim}{x})$, A_{ij}, or $\varphi(t, t')$ depend on additional parameters. We will be chiefly concerned with the case where $F(t, \underset{\sim}{x})$ depends on $\underset{\sim}{\lambda}$, but neither A_{ij} nor $\varphi(t, t')$ do so. The approach to be taken will reflect the effect of such additional unknown parameters on the Bayes estimates of $\underset{\sim}{\alpha}$ and will also consider the question of optimum estimation of the components of $\underset{\sim}{\lambda}$ itself.

In general, let

$$p(\underset{\sim}{S}, \underset{\sim}{\alpha}, \underset{\sim}{\lambda}) = \text{joint a priori p.d.f. of } \underset{\sim}{S}, \underset{\sim}{\alpha}, \text{ and } \underset{\sim}{\lambda} \tag{17}$$

$$p(\underset{\sim}{S}, \underset{\sim}{\alpha}, \underset{\sim}{\lambda}) = p(\underset{\sim}{S} \mid \underset{\sim}{\alpha}, \underset{\sim}{\lambda})\, p(\underset{\sim}{\alpha} \mid \underset{\sim}{\lambda})\, q(\underset{\sim}{\lambda}) \tag{18}$$

Here, $\underset{\sim}{S}$ is considered finite-dimensional. The function $q(\underset{\sim}{\lambda})$ is the a priori p.d.f. of $\underset{\sim}{\lambda}$.

Now, define

$$p(\underset{\sim}{\alpha}, \underset{\sim}{\lambda} \mid \underset{\sim}{S}) = \text{joint a posteriori p.d.f. of } (\underset{\sim}{\alpha}, \underset{\sim}{\lambda}) \text{ given } \underset{\sim}{S} \tag{19}$$

$$= \frac{p(\underset{\sim}{S}, \underset{\sim}{\alpha}, \underset{\sim}{\lambda})}{\int\int p(\underset{\sim}{S}, \underset{\sim}{\alpha}, \underset{\sim}{\lambda})\, d\underset{\sim}{\alpha}\, d\underset{\sim}{\lambda}}$$

$$p(\underset{\sim}{\alpha} \mid \underset{\sim}{S}) = \text{marginal a posteriori p.d.f. of } \underset{\sim}{\alpha} \text{ given } \underset{\sim}{S} \tag{20}$$

$$= \int p(\underset{\sim}{\alpha}, \underset{\sim}{\lambda} \mid \underset{\sim}{S})\, d\underset{\sim}{\lambda}$$

$$p(\underset{\sim}{\alpha} \mid \underset{\sim}{S}, \underset{\sim}{\lambda}) = \text{conditional a posteriori p.d.f. of } \underset{\sim}{\alpha} \text{ given } \underset{\sim}{S} \text{ and } \underset{\sim}{\lambda} \tag{21}$$

$$= \frac{p(\underset{\sim}{S} \mid \underset{\sim}{\alpha}, \underset{\sim}{\lambda})\, p(\underset{\sim}{\alpha} \mid \underset{\sim}{\lambda})}{\int p(\underset{\sim}{S} \mid \underset{\sim}{\alpha}, \underset{\sim}{\lambda})\, p(\underset{\sim}{\alpha} \mid \underset{\sim}{\lambda})\, d\underset{\sim}{\alpha}}$$

$$q(\underset{\sim}{S} \mid \underset{\sim}{\lambda}) = \text{conditional p.d.f. of } \underset{\sim}{S} \text{ given } \underset{\sim}{\lambda} \tag{22}$$

$$= \int p(\underset{\sim}{S} \mid \underset{\sim}{\alpha}, \underset{\sim}{\lambda})\, p(\underset{\sim}{\alpha} \mid \underset{\sim}{\lambda})\, d\underset{\sim}{\alpha}$$

$$q(\boldsymbol{\lambda} \mid \mathbf{S}) = \text{a posteriori marginal p.d.f. of } \boldsymbol{\lambda} \text{ given } \mathbf{S} \tag{23}$$

$$= \int p(\boldsymbol{\alpha}, \boldsymbol{\lambda} \mid \mathbf{S}) \, d\boldsymbol{\alpha}$$

$$= \frac{q(\mathbf{S} \mid \boldsymbol{\lambda}) \, q(\boldsymbol{\lambda})}{\int q(\mathbf{S} \mid \boldsymbol{\lambda}) \, q(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda}}$$

$$\hat{\beta}(\lambda) = \text{conditional Bayes estimate of } \beta \text{ given } \boldsymbol{\lambda} \tag{24}$$

$$= \int \beta(\boldsymbol{\alpha}) \, p(\boldsymbol{\alpha} \mid \mathbf{S}, \boldsymbol{\lambda}) \, d\boldsymbol{\alpha}$$

$$\hat{\beta} = \text{overall Bayes optimum estimate of } \beta(\boldsymbol{\alpha}) \tag{25}$$

$$= \int \beta(\boldsymbol{\alpha}) \, p(\boldsymbol{\alpha}, \boldsymbol{\lambda} \mid \mathbf{S}) \, d\boldsymbol{\alpha} \, d\boldsymbol{\lambda}$$

$$= \int \beta(\boldsymbol{\alpha}) \, p(\boldsymbol{\alpha} \mid \mathbf{S}) \, d\boldsymbol{\alpha}$$

Then, it is verified by direct substitution that

$$\hat{\beta} = \int \hat{\beta}(\boldsymbol{\lambda}) \, q(\boldsymbol{\lambda} \mid \mathbf{S}) \, d\boldsymbol{\lambda} \tag{26}$$

$$= \frac{\int \hat{\beta}(\boldsymbol{\lambda}) \, q(\mathbf{S} \mid \boldsymbol{\lambda}) \, q(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda}}{\int q(\mathbf{S} \mid \boldsymbol{\lambda}) \, q(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda}}$$

Equation (26) says that the overall Bayes estimate is the expected value of the conditional Bayes estimate, given $\underset{\sim}{\lambda}$, with respect to the marginal a posteriori p.d.f. of $\underset{\sim}{\lambda}$ given $\underset{\sim}{S}$. However, the function of $q(\underset{\sim}{S} \mid \underset{\sim}{\lambda})$ can be determined; this is done, in fact, by following the same procedure used to derive Eq. (5), carefully keeping track of all the terms involved.

The result is

$$q(\underset{\sim}{S} \mid \underset{\sim}{\lambda}) = \int p(\underset{\sim}{S} \mid \underset{\sim}{\alpha}, \underset{\sim}{\lambda})\, p(\underset{\sim}{\alpha} \mid \underset{\sim}{\lambda})\, d\underset{\sim}{\alpha} \tag{27}$$

$$= (\det \underset{\sim}{A}\, \underset{\sim}{B})^{-\frac{1}{2}} (2\pi)^{-\frac{N}{2}} (\det \underset{\sim}{\varphi})^{-\frac{1}{2}}$$

$$\times \exp\left\{-\tfrac{1}{2}\left[\underset{\sim}{S}^T \underset{\sim}{\eta}\, \underset{\sim}{S} - \underset{\sim}{S}^T \underset{\sim}{\eta}\, \underset{\sim}{F}\, \underset{\sim}{B}^{-1} \underset{\sim}{F}^T \underset{\sim}{\eta}\, \underset{\sim}{S}\right]\right\}$$

where $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{\varphi}$, $\underset{\sim}{\eta}$, $\underset{\sim}{F}$ may be functions of $\underset{\sim}{\lambda}$. Equation (27) applies to the case where the observation times are restricted to a set of N points $\{t_\mu\}$.

Let us now assume that $\underset{\sim}{\varphi}$, the covariance function of the additive noise, is independent of $\underset{\sim}{\lambda}$. Then, when Eq. (27) is substituted into Eq. (26), $(\det \underset{\sim}{\varphi})^{\frac{1}{2}}$ and $(2\pi)^{-\frac{N}{2}}$ drop out, so it is possible to go to the limit as $\{t_\mu\}$ becomes dense. Also, the following formula may be used:

$$\hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda}) = \underset{\sim}{B}^{-1}(\underset{\sim}{\lambda})\, \underset{\sim}{F}^T(\underset{\sim}{\lambda})\, \underset{\sim}{\eta}\, \underset{\sim}{S} \tag{28}$$

The result is

$$\hat{\beta} = \frac{\int \hat{\beta}(\underset{\sim}{\lambda}) \left[\det \underset{\sim}{A}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\right]^{-\frac{1}{2}} \exp\left[\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right] q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}}{\int \left[\det \underset{\sim}{A}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\right]^{-\frac{1}{2}} \exp\left[\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right] q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}} \tag{29}$$

In some special cases which will be of interest in later sections it is found that not only $\underset{\sim}{A}$ but also $\underset{\sim}{B}$ is independent of $\underset{\sim}{\lambda}$. In such cases

$$\hat{\beta} = \frac{\int \hat{\beta}(\underset{\sim}{\lambda}) \exp\left\{\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right\} q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}}{\int \exp\left\{\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right\} q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}} \tag{29a}$$

The expression Eq. (27) for $q(\underset{\sim}{S} \mid \underset{\sim}{\lambda})$ can also be used to determine Bayes optimum estimates for functionals of $\underset{\sim}{\lambda}$. Thus, suppose $\gamma(\underset{\sim}{\lambda})$ is any real-valued functional of $\underset{\sim}{\lambda}$. Then, if $\underset{\sim}{\eta}$ is independent of $\underset{\sim}{\lambda}$

$$\hat{\gamma} = \frac{\int \gamma(\underset{\sim}{\lambda}) \left[\det \underset{\sim}{A}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\right]^{-\frac{1}{2}} \exp\left[\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right] q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}}{\int \left[\det \underset{\sim}{A}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\right]^{-\frac{1}{2}} \exp\left[\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}(\underset{\sim}{\lambda})\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right] q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}} \tag{30}$$

If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are independent of $\underset{\sim}{\lambda}$

$$\hat{\gamma} = \frac{\int \gamma(\underset{\sim}{\lambda}) \exp\left[\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right] q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}}{\int \exp\left[\frac{1}{2}\, \hat{\underset{\sim}{\alpha}}^{T}(\underset{\sim}{\lambda})\ \underset{\sim}{B}\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda})\right] q(\underset{\sim}{\lambda})\ d\underset{\sim}{\lambda}} \tag{30a}$$

C. ALTERNATIVE METHODS OF EVALUATING $\hat{\underset{\sim}{\alpha}}$ and $\hat{\beta}$

An alternative method for evaluating $\hat{\underset{\sim}{\alpha}}$ and $\hat{\beta}$ for various functionals of $\underset{\sim}{\alpha}$ can be obtained from linear optimum signal estimation theory. Starting with Eq. (1), suppose the goal is to estimate $\alpha(\underset{\sim}{x}^{(i)})$, given $S(t)$. The correlation function of $\alpha(\underset{\sim}{x}^{(i)})$ with $S(t)$ is

$$E\left[\alpha(\underset{\sim}{x}^{(i)})\ S(t)\right] = \sum_j A_{ij}\ F(t, \underset{\sim}{x}^{(j)}) \tag{31}$$

The autocovariance function of $S(t)$ is

$$\varphi^*(t, t') = E\left[S(t)\ S(t')\right] \tag{32}$$

$$= \varphi(t, t') + \sum_{j,k} A_{jk}\ F(t, \underset{\sim}{x}^{(j)})\ F(t', \underset{\sim}{x}^{(k)})$$

Then, the Bayes optimum estimate $\hat{\alpha}(\underset{\sim}{x}^{(i)})$ is[4]

$$\hat{\alpha}(\underset{\sim}{x}^{(i)}) = \sum_j A_{ij} \lim \sum_{\mu,\nu} \eta^*_{\mu\nu}\ F(t_\mu, \underset{\sim}{x}^{(j)})\ S(t_\nu) \tag{33}$$

where

$$\eta^*_{\mu\nu} = \mu,\ \nu^{\text{th}} \text{ element of matrix inverse to } \varphi^*(t_\mu, t_\nu) \tag{34}$$

This may be written

$$\underset{\sim}{\varphi}^* = \underset{\sim}{\varphi} + \underset{\sim}{F}\ \underset{\sim}{A}\ \underset{\sim}{F}^T \tag{35}$$

$$\underset{\sim}{\eta}^* = (\underset{\sim}{\varphi}^*)^{-1} \tag{36}$$

$$\hat{\underset{\sim}{\alpha}} = \underset{\sim}{A}\ \underset{\sim}{F}^T\ \underset{\sim}{\eta}^*\ \underset{\sim}{S} \tag{37}$$

In this form, it is immediately apparent how these expressions may be generalized when the sum in Eq. (1) is replaced by an integral. Specifically, suppose

$$S(t) = \int_X \alpha(\underset{\sim}{x})\ F(t, \underset{\sim}{x})\ d\underset{\sim}{x} + \varepsilon(t) \tag{1a}$$

and let

$$A(\underset{\sim}{x}, \underset{\sim}{x}') = E\left[\alpha(\underset{\sim}{x})\ \alpha(\underset{\sim}{x}')\right] \tag{2a}$$

Then, Eqs. (35)-(37) still hold, but with

$$\underset{\sim}{F}\ \underset{\sim}{A}\ \underset{\sim}{F}^T = \int_X \int_X F(t, \underset{\sim}{x})\ A(x, \underset{\sim}{x}')\ F(t', \underset{\sim}{x}')\ d\underset{\sim}{x}\ d\underset{\sim}{x}' \tag{38}$$

$$\underset{\sim}{A}\ \underset{\sim}{F}^T = \int_X A(\underset{\sim}{x}, \underset{\sim}{x}')\ F(t, \underset{\sim}{x}')\ d\underset{\sim}{x}' \tag{39}$$

and so forth.

Still another alternative expression for $\hat{\underset{\sim}{\alpha}}$ can be stated for the cases represented by Eq. (1), using the finite sum expression, when $\underset{\sim}{A}$ is a diagonal matrix. Suppose it is desired to estimate $\alpha(\underset{\sim}{x}^{(i)})$. The sum over $j \neq i$ can be treated as additional noise. Thus, the problem as given by Eq. (1) can be replaced by an equivalent one in which

$$S(t) = \alpha(\underset{\sim}{x}^{(i)})\ F(t, \underset{\sim}{x}^{(i)}) + \tilde{\varepsilon}(t) \tag{1b}$$

where

$$\tilde{\varphi}_i(t, t') = E\left[\tilde{\varepsilon}(t)\ \tilde{\varepsilon}(t')\right] = \varphi(t, t') + \sum_{j \neq i} A_{jj}\ F(t, \underset{\sim}{x}^{(j)})\ F(t', \underset{\sim}{x}^{(j)}) \tag{40}$$

This gives rise to an equivalent problem in which the matrix $\tilde{B}_i$ is a scalar. The formula for $\hat{\alpha}(x^{(i)})$ becomes

$$\hat{\alpha}(x^{(i)}) = \frac{\int\int F(t, x^{(i)})\, \tilde{\eta}_i(t, t')\, S(t')\, dt\, dt'}{(A_{ii})^{-1} + \int\int F(t, x^{(i)})\, \tilde{\eta}_i(t, t')\, F(t', x^{(i)})\, dt\, dt'} \tag{41}$$

where the integrals are symbolic representations of limits of finite sums over μ, ν of the kind used above.

In many cases, the matrices $\tilde{\varphi}_i$ are virtually independent of i (i.e., the addition of terms involving i to the sum in Eq. (40) would be negligible); in such cases, the result would be

$$\tilde{\varphi} \approx \varphi + F\, A\, F^T \tag{42}$$

and

$$\hat{\alpha}(x^{(i)}) = \frac{(F^T\, \tilde{\eta}\, S)_i}{a(x^{(i)}) + (F^T\, \tilde{\eta}\, F)_{ii}} \tag{43}$$

where a is defined

$$a(x^{(i)}) = \left[A(x^{(i)}, x^{(i)})\right]^{-1} \tag{44}$$

Both the expressions for $\hat{\alpha}$ given here and those given in Section II.A are useful. Those in Section II.A are most useful if the scattering body can be represented as relatively few scatterers (more precisely, if there are relatively few positions $x^{(i)}$ which are candidates for the position of a major scattering center); or if $(F^T\, \eta\, F + A^{-1})$ is for other reasons relatively easy to invert. On the other hand, the expressions in the present subsection are useful

if $\underset{\sim}{\varphi} + \underset{\sim}{F}\,\underset{\sim}{A}\,\underset{\sim}{F}^T$ is relatively easy to invert (say, if the total time of observation is very short).

Finally, it is of interest to discuss alternative evaluations of $\hat{\beta}_Q$ and the manner in which these generalize to the case in which S(t) is given by an integral expression of the form Eq. (1a).

First, when S(t) is given by a finite sum, $\hat{\beta}_Q$ is given by Eq. (16). Either of the expressions Eq. (9) or (37) can be used for $\hat{\underset{\sim}{\alpha}}$ (if $\underset{\sim}{A}$ is diagonal, Eq. (41) can also be used), but $\underset{\sim}{B}$ is still given by Eq. (10). Also, if $\underset{\sim}{Q}$ depends on only one point $\underset{\sim}{x}^{(i)}$ and $\underset{\sim}{A}$ is diagonal, then $\underset{\sim}{B}$ in Eq. (16) can be replaced by a scalar:

$$\tilde{B}_i = \frac{1}{A_{ii}} + F_i^T\,\tilde{\eta}_i\,F_i^T \tag{45}$$

The manner in which the expression Eq. (16) for $\tilde{\beta}_Q$ generalizes to the cases where S(t) is given by the integral expression Eq. (1a) is not completely obvious. Here, β_Q is defined

$$\beta_Q = \int\int \alpha(\underset{\sim}{x})\; Q(\underset{\sim}{x}, \underset{\sim}{x}')\; \alpha(\underset{\sim}{x}')\; d\underset{\sim}{x}\; d\underset{\sim}{x}' \tag{14a}$$

There is no difficulty in generalizing the term $\hat{\underset{\sim}{\alpha}}^T\,\underset{\sim}{Q}\,\hat{\underset{\sim}{\alpha}}$ in Eq. (16)--this is simply carried over with $\hat{\underset{\sim}{\alpha}}$ given by Eq. (37). The problem is in deciding how to generalize $\underset{\sim}{B}^{-1}$ in Eq. (16).

This problem can be approached as follows. Comparing two expressions for $\hat{\underset{\sim}{\alpha}}$, Eqs. (9) and (37), it can be seen that

$$\underset{\sim}{B}^{-1}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta} = \underset{\sim}{A}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta}^* \tag{46}$$

Thus

$$\underset{\sim}{B}^{-1} = \underset{\sim}{A}\,\underset{\sim}{F}^{T}\,\underset{\sim}{\eta}^{*}\,\underset{\sim}{F}(\underset{\sim}{F}^{T}\,\underset{\sim}{\eta}\,\underset{\sim}{F})^{-1} \tag{47}$$

In this form, the expression generalizes to the cases represented by the integral expression Eq. (1a). The interpretation of $\underset{\sim}{B}^{-1}$ in such cases is

$$B^{-1}(x, x') = \int A(x, y)\, F(t, y)\, \eta^{*}(t, t') \times F(t', z)\, U(z, x')\, dy\, dz\, dt\, dt' \tag{48}$$

where

$$U^{-1}(x, x') = \int F(t, x)\, \eta(t, t')\, F(t', x')\, dt\, dt' \tag{49}$$

These are all symbolic expressions for operators.

Expressions for $\hat{\beta}$ when β is a higher order form in $\underset{\sim}{\alpha}$ can similarly be developed for the integral case represented by Eq. (1a). One can regard $\underset{\sim}{B}^{-1}$, given by Eq. (47), as the covariance function of the (Gaussian) conditional distribution of $\underset{\sim}{\alpha}$ given $\underset{\sim}{S}$; and $\hat{\underset{\sim}{\alpha}}$, given by Eq. (37), as the mean-value function. The expressions for $\hat{\beta}$ for higher order forms are then obtained using the well-known expressions for the higher order moments of a Gaussian random process in terms of its means and covariance function.

D. MOMENTS OF ESTIMATION ERRORS

Often the problem of primary interest is to determine the mean square estimation error. Of course, given the expressions for the estimates themselves, the mean square error can be estimated by

Monte Carlo techniques. It is much preferable, however, to use analytic expressions.

Such expressions can be stated for linear and quadratic functionals of $\underset{\sim}{\alpha}$; it is easy to see how they could also be stated for higher order functionals by extending the same techniques in an obvious, though tedious, manner. These statements are made assuming no dependence of $\underset{\sim}{F}$ on additional unknown parameters. If such dependence exists, the problem of writing expressions for mean square error is much more complicated; of course, lower bounds can be obtained by using the expressions which assume the unknown parameters to be known (or by using Cramer-Rao inequalities).

The mean square error expressions of interest are the conditional mean square error, given that $\underset{\sim}{\alpha}$ has a definite value $\underset{\sim}{\alpha}_o$; and the unconditional mean square error, given an arbitrary "true" statistical distribution of $\underset{\sim}{\alpha}$. Cases of interest must include those in which the "true" statistical distribution of $\underset{\sim}{\alpha}$ differs from the a priori distribution assumed for purposes of forming the Bayes estimate, although a special case of interest arises when these two distributions coincide. It may also be of interest to determine mean square error when the true covariance of the noise $\{\epsilon(t)\}$ differs from that assumed in forming the Bayes estimate.

First consider the means and second moments of $\hat{\beta}_L$ and $\hat{\beta}_Q$ conditional on $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}_o$. Recall that

$$\underset{\sim}{S} = \underset{\sim}{F}\,\underset{\sim}{\alpha}_o + \underset{\sim}{\epsilon} \tag{50}$$

Thus, using the formulas of Section II.A for $\hat{\underset{\sim}{\alpha}}$,

$$E\left[\hat{\beta}_L \mid \underset{\sim}{\alpha}_o\right] = \underset{\sim}{L}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta}\,\underset{\sim}{F}\,\underset{\sim}{\alpha}_o \tag{51}$$

and

$$E\left[(\hat{\beta}_L - \beta_L)^2 \mid \underset{\sim}{\alpha}_o\right] = \underset{\sim}{L}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta}\,\underset{\sim}{\varphi}_o\,\underset{\sim}{\eta}\,\underset{\sim}{F}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{L}^T + \left[\underset{\sim}{L}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta}\,\underset{\sim}{F}\,\underset{\sim}{\alpha}_o - \beta(\underset{\sim}{\alpha}_o)\right]^2 \tag{52}$$

Here, $\varphi_o(t, t')$ is the "actual" covariance of $\underset{\sim}{\epsilon}$; of course, if this is also the covariance assumed in the Bayes estimation, then $\underset{\sim}{\eta}\,\underset{\sim}{\varphi}_o\,\underset{\sim}{\eta} = \underset{\sim}{\eta}$.

If the assumed a priori distribution of $\underset{\sim}{\alpha}$ has very large variances, i.e., if the estimates depend mainly on the observed signal and only weakly on the a priori mean values, then $\underset{\sim}{B}^{-1}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta}\,\underset{\sim}{F} \approx \underset{\sim}{I}$ and the second term in Eq. (52) vanishes, while the right side of Eq. (51) becomes simply $\beta(\underset{\sim}{\alpha}_o)$.

The moments of $\hat{\beta}_Q$ conditional on $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}_o$ can best be determined as follows. First

$$\hat{\beta}_Q - \beta_Q(\underset{\sim}{\alpha}_o) = \underset{\sim}{\epsilon}^T\,\underset{\sim}{M}\,\underset{\sim}{\epsilon} + \underset{\sim}{R}\,\underset{\sim}{\epsilon} + W \tag{53}$$

where

$$\underset{\sim}{M} = \underset{\sim}{\eta}\,\underset{\sim}{F}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{Q}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{F}^T\,\underset{\sim}{\eta} \tag{54}$$

$$\underset{\sim}{R} = 2\,\underset{\sim}{\alpha}_o^T\,\underset{\sim}{F}^T\,\underset{\sim}{M} \tag{55}$$

$$W = \underset{\sim}{\alpha}_o^T\,\underset{\sim}{F}^T\,\underset{\sim}{M}\,\underset{\sim}{F}\,\underset{\sim}{\alpha}_o - \beta_Q(\underset{\sim}{\alpha}_o) + \text{Trace}\left[\underset{\sim}{B}^{-1}\,\underset{\sim}{Q}\right] \tag{56}$$

Therefore,

$$E\left[\hat{\beta}_Q \mid \underset{\sim}{\alpha}_o\right] = \text{Trace}\left[\underset{\sim}{B}^{-1}\,\underset{\sim}{Q}\right] + \underset{\sim}{\alpha}_o^T\,\underset{\sim}{F}\,\underset{\sim}{M}\,\underset{\sim}{F}\,\underset{\sim}{\alpha}_o \qquad (57)$$
$$+ \text{Trace}\left[\underset{\sim}{M}\,\underset{\sim}{\varphi}_o\right]$$

and

$$E\left[(\hat{\beta}_Q - \beta_Q)^2 \mid \underset{\sim}{\alpha}_o\right] = E\left[(\underset{\sim}{\varepsilon}^T\,\underset{\sim}{M}\,\underset{\sim}{\varepsilon})^2 \mid \underset{\sim}{\alpha}_o\right] \qquad (58)$$
$$+ E\left[(\underset{\sim}{R}\,\underset{\sim}{\varepsilon})^2 \mid \underset{\sim}{\alpha}_o\right]$$
$$+ 2\,W\,E\left[\underset{\sim}{\varepsilon}^T\,\underset{\sim}{M}\,\underset{\sim}{\varepsilon} \mid \underset{\sim}{\alpha}_o\right]$$

Using standard formulas for fourth order moments of Gaussian processes,

$$E\left[(\hat{\beta}_Q - \beta_Q)^2 \mid \underset{\sim}{\alpha}_o\right] = \left[\text{Trace}\,\underset{\sim}{M}\,\underset{\sim}{\varphi}_o\right]^2 \qquad (59)$$
$$+ 2\,\text{Trace}\,(\underset{\sim}{M}\,\underset{\sim}{\varphi}_o\,\underset{\sim}{M}\,\underset{\sim}{\varphi}_o)$$
$$+ \underset{\sim}{R}\,\underset{\sim}{\varphi}_o\,\underset{\sim}{R}^T$$
$$+ 2\,W\,\text{Trace}\left[\underset{\sim}{M}\,\underset{\sim}{\varphi}_o\right]$$

Now, suppose it is desired to obtain the unconditional expectations with respect to some given "actual" statistical distribution of $\underset{\sim}{\alpha}_o$, which can be denoted $dP(\underset{\sim}{\alpha}_o)$. All such moments are simply the integrals of the conditional expectations with respect to $dP(\underset{\sim}{\alpha}_o)$. If $dP(\underset{\sim}{\alpha}_o)$ is Gaussian with covariance $\underset{\sim}{A}_o$, then the procedure to be followed is similar to that for determining the conditional moments. Thus, all the conditional moments above can be expressed as the sum of constants and linear and quadratic forms in $\underset{\sim}{\alpha}_o$. The moments of all these can be expressed in terms of the matrix $\underset{\sim}{A}_o$. For example

$$E\left[(\hat{\beta}_Q - \beta_Q)^2\right] = \left[\mathrm{Trace}\,(\underset{\sim}{M}\,\underset{\sim}{\varphi}_o)\right]^2 + 2\,\mathrm{Trace}(\underset{\sim}{M}\,\underset{\sim}{\varphi}_o\,\underset{\sim}{M}\,\underset{\sim}{\varphi}_o) + 2\,\mathrm{Trace}(\underset{\sim}{M}\,\underset{\sim}{\varphi}_o)\,\mathrm{Trace}(\underset{\sim}{B}^{-1}\,\underset{\sim}{Q} + 4\,\mathrm{Trace}\left[\underset{\sim}{A}_o\,\underset{\sim}{F}^T\,\underset{\sim}{M}\,\underset{\sim}{\varphi}_o\,\underset{\sim}{M}\,\underset{\sim}{F}\right] + \mathrm{Trace}\left[\underset{\sim}{A}_o(\underset{\sim}{F}^T\,\underset{\sim}{M}\,\underset{\sim}{F} - \underset{\sim}{Q})\right] \tag{60}$$

These expressions all can be generalized to the case where $\hat{\underset{\sim}{\alpha}}$ is given by Eq. (37), and hence to the case where S(t) is given by the integral Eq. (1a), simply by interpreting $\underset{\sim}{B}^{-1}$ in all the above expressions via Eq. (47).

These techniques can obviously be extended to determine the characteristic functions (and hence the p.d.f.'s) of the estimates $\hat{\beta}_I$ and $\hat{\beta}_Q$, as well as the moments of estimates of higher order forms in $\underset{\sim}{\alpha}$. However, the necessary algebra becomes extremely tedious.

E. ADDITIONAL REMARKS

It is clear how the function $Q(\underset{\sim}{x}, \underset{\sim}{x}')$ can be chosen so as to yield information about the integrated radar cross section contributed by specified sections of the target, as well as to make estimates of body dimensions (e.g., by letting $Q(\underset{\sim}{x}, \underset{\sim}{x}') = |\underset{\sim}{x}|^a\,\delta(\underset{\sim}{x}' - \underset{\sim}{x})$, where a is a large integer). It will also be of interest in this connection to form estimates of other functions of β_Q, or of functions

depending on two quadratic functionals. For example, functions of the form $\left[\beta_Q\right]^{1/m}$ or β_{Q_1}/β_{Q_2} or, more generally, $f\left[\beta_{Q_1}, \beta_{Q_2}\right]$ may be of interest.

In many such cases, even though a strong signal assumption may not be justified in determining $\hat{\beta}_Q$ from $\hat{\underset{\sim}{\alpha}}$, such an assumption may be justified for determining $\hat{f}$ by $\hat{f} = f\left(\hat{\beta}_{Q_1}, \hat{\beta}_{Q_2}\right)$. An error analysis can then be carried out by expanding $f\left(\hat{\beta}_{Q_1}, \hat{\beta}_{Q_2}\right)$ to first order in the error terms $\hat{\beta}_{Q_1} - \beta_{Q_1}$, $\hat{\beta}_{Q_2} - \beta_{Q_2}$ and then applying the error formulas for $\hat{\beta}_{Q_1}$ and $\hat{\beta}_{Q_2}$. (This would require cross-moments of $\hat{\beta}_{Q_1}$ and $\hat{\beta}_{Q_2}$, which can be determined by the same sort of technique as used in Section II.D.)

A possibly useful method of evaluating the expressions for $\hat{\underset{\sim}{\alpha}}$, $\hat{\beta}_L$, $\hat{\beta}_Q$, and so forth, alternative to those stated above, is the use of recursive Bayes estimates.(5,6) The application of recursive methods to the problems considered in this Memorandum is currently under investigation.

III. APPLICATION TO ROTATING TARGETS

A. TARGET MODELS AND SIGNAL MODELS

The target and signal models to be considered can be represented in the following way (omitting the noise term $\epsilon(t)$)

$$S(t) = \sum_i a(r_i, \varphi_i) \cos\left[\omega_c t + r_i \sin(\omega_r t + \varphi_i) + \xi_i\right] \tag{61}$$

where

ω_c = carrier frequency (radians/sec)

ω_r = target rotation frequency (radians/sec)

ξ_i = RF phase parameter of signal from i^{th} scatterer (further discussed below)

(r_i, φ_i) = location of i^{th} scatterer in a coordinate system fixed to the rotating body (further described below)

$a(r_i, \varphi_i)$ = amplitude (absolute magnitude) of i^{th} scatterer

In what follows, φ always represents an angular coordinate unless otherwise specifically stated and should not be confused with the covariance function φ of the observation noise.

The coordinate system (r, φ) is defined as follows. Initially, suppose that the axis of rotation is normal to the line of sight to the radar. Then, the (r, φ)-plane is normal to the rotation axis and contains the line of sight. The coordinates (r, φ) are then cylindrical coordinates of a point on the target in a coordinate system fixed to the body, in which the axis of rotation is the z-axis. At any time t, the angle of the $\varphi = 0$ axis with respect to a plane normal to the line of sight is $\omega_r t$. (In other words, $\omega_r t + \pi/2$ is the angle between the $\varphi = 0$ axis and the line of sight.) The units

of r are chosen so that one wavelength equals 4π units. Otherwise a factor $4\pi/\lambda$ would have to multiply r in Eq. (61).

The same model can be applied to some types of targets even when the axis of rotation is not normal to the line of sight. Specifically, suppose all the significant scatterers on the body are known a priori to lie in a plane which is normal to the rotation axis. In such a case, let (r, φ) still be cylindrical coordinates fixed to the body, with the z-axis being the rotation axis. The signal can still be described by Eq. (61) if r_i is replaced by $r_i \sin \Psi$, where Ψ is the angle between the rotation axis and the line of sight. In this case, $\omega_r t$ is the angle at time t between the $\varphi = 0$ axis and the line of intersection of the plane normal to the line of sight and the (r, φ) plane fixed to the target.)

The symbol ξ_i denotes an additional unknown phase parameter associated with the signal reflected from the i^{th} scatterer. It is assumed to be uniformly distributed over $(0, 2\pi)$. Thus, Eq. (61) can also be written

$$\begin{aligned} S(t) &= \sum_i \alpha(r_i, \varphi_i, 1) \sin\left[\omega_c t + r_i \sin(\omega_r t + \varphi_i)\right] \\ &\quad + \sum_i \alpha(r_i, \varphi_i, 2) \cos\left[\omega_c t + r_i \sin(\omega_r t + \varphi_i)\right] \end{aligned} \tag{62}$$

where

$$\begin{aligned} \alpha(r_i, \varphi_i, 1) &= a(r_i, \varphi_i) \cos \xi_i \\ \alpha(r_i, \varphi_i, 2) &= a(r_i, \varphi_i) \sin \xi_i \end{aligned} \tag{63}$$

Thus, the superscript i of Section II is here a pair of indices (i, μ) where i indexes the scatterers and μ takes on only two values, 1 and 2. Also

$$\underset{\sim}{x} = (r, \varphi, \mu) \tag{64}$$

where $\mu = 1$ or 2. It is assumed that the components of $\underset{\sim}{\alpha}$ have a priori a joint Gaussian distribution.

Since there are no modulating functions multiplying the sum in Eq. (61), CW illumination of the target is in effect being assumed. However, the same model can also represent the case of a pulse radar for which the range resolution of a pulse is much larger than the physical extent of the target, and for which the pulse repetition frequency is sufficiently high that the maximum distance moved by any scatterer in the interpulse period is small compared with a wavelength.

It will be assumed that ω_r is known a priori. This is not an unrealistic assumption in many cases (e.g., when the target is known a priori to be stabilized and the effective rotation is due to non-radial translatory motion of the target with respect to the radar); also, it can often be assumed that the rotation rate has been accurately established from prior observations, since rotation rate is a parameter often capable of being very accurately established.

It is clear that the above model implies that a coherent radar is used. This does not imply that signal phase is known a priori, but merely that the RF phase of the received signal plus noise is observed

and made use of. An interesting problem is to determine the effect of lack of coherence, either in the radar processing or because of phase errors. It is known that much signature analysis can be done even with completely incoherent operation (e.g., even if the radar measures only signal amplitude). The discussion of this question is given in Section III.E below.

The targets envisaged by the representation stated in Eq. (61) consist of sets of rigidly connected point scatterers. The potential positions of the point scatterers with respect to the rotation axis are assumed to be known a priori to within approximately one-half wavelength (one wavelength in the two-way path). The additional unknown parameters ξ_i are included to allow the precise position of the i^{th} scatterer to differ from the position (r_i, φ_i) by up to one-half wavelength. (For some further cases treated below, it is required that all the ξ_i's be equal or highly correlated.) It is necessary to assume that the maximum separation between potential positions (r_i, φ_i) of scatterers is of the order of one-half wavelength. Strictly speaking, (r_i, φ_i) should be called the i^{th} mean potential scatterer position.

It is somewhat easier to visualize the potential scatterer positions, projected on the (r, φ)-plane, as a regular rectangular lattice, with

$$\begin{aligned} i^{th} \text{ position} &= (u_m, v_n) \\ i &\sim (m, n) \\ u_m &= r_i \cos \varphi_i \\ v_n &= r_i \sin \varphi_i \end{aligned} \tag{65}$$

On occasion the corresponding integral form over the (r, φ)-plane will be used.

The scattering model just stated also applies to sets of rigidly connected spheres, even when the spherical radii are larger than a wavelength.* In such cases, the scattering point on each sphere is, at any instant, the point on the sphere's surface intersected by the line connecting the center of the sphere to the radar. If the center of a given sphere describes a circle of radius r, so also does the instantaneous scattering point, but the circle is displaced toward the radar by the radius of the sphere. If the scattering spheres have different radii, this is accounted for by the phase variables ξ_i.

In the models to be treated, it is assumed that each scatterer is visible during the whole observation interval. This might be true if the target scatterers were all located on a rotating disk, or on a rotating line, with slightly tilted rotation axis; or if the target otherwise consisted of individual points which did not shadow each other. A similar model might apply to long thin bodies which are sufficiently symmetrical about the lengthwise axis.

In many cases, one is interested in bodies for which, even if they can be reasonably modeled as collections of rigidly connected point scatterers, each scatterer is visible only part of the time, i.e., is visible over some restricted range of values of $\varphi + \omega_r t$, as on the surface of a rotating sphere. It will be apparent that the treatment to be given can be generalized to such cases, but detailed evaluations are much harder and will only be indicated.

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*It should be remembered that (r_i, φ_i) are possible scatterer locations; in any actual case, many of these possible locations might be empty, that is, occupied by zero amplitude scatterers.

In all cases, it will be assumed that the observation noise is "white":

$$\varphi(t,\ t') = \frac{N_o}{2}\,\delta(t - t') \tag{66}$$

$$\eta(t,\ t') = \frac{2}{N_o}\,\delta(t - t') \tag{67}$$

Two categories of target models will be treated: first, a model in which the potential scatterer positions are not assumed a priori to be collinear; and second, a model in which it is assumed that the scatterer positions are known a priori to be collinear.

B. NONCOLLINEAR A PRIORI SCATTERER DISTRIBUTIONS

For noncollinear a priori scatterer distributions, the matrix $\underset{\sim}{A}$ will be taken to be diagonal:

$$A(r_i,\ \varphi_i,\ \mu,\ r_j,\ \varphi_j,\ \mu') = A_i\ \delta_{i\mu}^{j\mu'} \tag{68}$$

where δ is the Kronecker delta. The expressions necessary to evaluate the Bayes estimates can then be derived by either of two approaches.

First Approach

The first approach will be to calculate $B(\underset{\sim}{x},\ \underset{\sim}{x}')$ by application of the formula Eq. (7). This requires evaluation of the following types of integrals:

$$I_{11} = \int \sin\left[\omega_c\ t + r\ \sin\ (\omega_r\ t + \varphi)\right] \sin\left[\omega_c\ t + r'\ \sin\ (\omega_r\ t + \varphi')\right] dt \tag{69}$$

$$I_{12} = \int \sin\left[\omega_c\ t + r\ \sin\ (\omega_r\ t + \varphi)\right] \cos\left[\omega_c\ t + r'\ \sin\ (\omega_r\ t + \varphi')\right] dt \tag{70}$$

$$I_{22} = \int \cos\left[\omega_c\ t + r\ \sin\ (\omega_r\ t + \varphi)\right] \cos\left[\omega_c\ t + r'\ \sin\ (\omega_r\ t + \varphi')\right] dt \tag{71}$$

$$I_{21} = \text{same as } I_{12} \text{ with } r,\ \varphi \text{ and } r',\ \varphi' \text{ interchanged} \tag{72}$$

The integrals can be assumed to extend over an interval $|\theta| = |\omega_r t| \leq \theta_M$; without loss of generality it can be assumed that $t = 0$ is the midpoint of the observation interval.

If ω_c is sufficiently large, as it will be in virtually all cases of interest, then

$$I_{11} = I_{22} = \frac{1}{2\omega_r} \int \cos \left[|\rho| \sin (\theta + \theta^*) \right] d\theta \tag{73}$$

$$I_{12} = \frac{1}{2\omega_r} \int \sin \left[|\rho| \sin (\theta + \theta^*) \right] d\theta \tag{74}$$

$$I_{21} = \frac{1}{2\omega_r} \int \sin \left[|\rho| \sin (\theta - \theta^*) \right] d\theta \tag{75}$$

where

$$|\rho| = \left[r^2 + r'^2 - 2r\, r' \cos (\varphi - \varphi') \right]^{\frac{1}{2}} \tag{76}$$

$$\theta^* = \tan^{-1} \left[\frac{r \sin \varphi - r' \sin \varphi'}{r \cos \varphi - r' \cos \varphi'} \right] \tag{77}$$

Now, the integrands can be expanded

$$\cos \left[|\rho| \sin (\theta + \theta^*) \right]$$

$$= J_o \left(|\rho| \right) + 2 \sum_{n=1}^{\infty} J_{2n} \left(|\rho| \cos \left[2n(\theta + \theta^*) \right] \right. \tag{78}$$

$$\sin \left[|\rho| \sin (\theta + \theta^*) \right] = 2 \sum_{n=0}^{\infty} J_{2n+1} \left(|\rho| \right) \sin \left[(2n + 1)(\theta + \theta^*) \right] \tag{79}$$

Consequently, recalling that the integrals in $\theta = \omega_r t$ are assumed to extend from $\theta = -\theta_M$ to $\theta = +\theta_M$,

$$I_{11} = I_{22} = \frac{\theta_M}{\omega_r} J_o\left(|\rho|\right) + \frac{2}{\omega_r} \sum_{n=1}^{\infty} J_{2n}\left(|\rho|\right)\left[\frac{\cos 2n\,\theta^* \sin 2n\,\theta_M}{n}\right] \tag{80}$$

$$I_{12} = -I_{21} = \frac{1}{\omega_r} \sum_{n=0}^{\infty} J_{2n+1}\left(|\rho|\right)\left[\frac{\sin (2n+1)\,\theta^* \sin (2n+1)\,\theta_M}{2n+1}\right] \tag{81}$$

If $\theta_M = K\pi$, where K is an integer, which is to say, if the observation interval is an integral number of periods of the rotation, then

$$I_{11} = I_{22} = \frac{K\pi}{\omega_r} J_o\left(|\rho|\right) \tag{82}$$

$$I_{12} = -I_{21} = 0 \tag{83}$$

If $\theta_M = \frac{K\pi}{2}$, with K an integer, then

$$I_{11} = I_{22} = \frac{K\pi}{2\omega_r} J_o\left(|\rho|\right) \tag{84}$$

$$I_{12} = -I_{21} = \frac{1}{\omega_r} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sin\left[(2n+1)\,\theta^*\right] J_{2n+1}\left(|\rho|\right) \tag{85}$$

The formula for $B(\underset{\sim}{x}, \underset{\sim}{x}')$ in terms of I_{11}, etc., is

$$B(r, \varphi, 1, r', \varphi', 1) = B(r, \varphi, 2, r', \varphi', 2) \tag{86}$$

$$= \left[A(r, \varphi)\right]^{-1} + \frac{2}{N_o} I_{11}(r, \varphi, r, \varphi), \text{ if } (r, \varphi) = (r', \varphi')$$

$$= \frac{2}{N_o} I_{11}(r, \varphi, r', \varphi'), \text{ if } (r, \varphi) \neq (r', \varphi')$$

$$B(r, \varphi, 1, r', \varphi', 2) = \frac{2}{N_o} I_{12}(r, \varphi, r', \varphi') \tag{87}$$

$$B(r, \varphi, 2, r', \varphi', 1) = \frac{2}{N_o} I_{21}(r, \varphi, r', \varphi') \tag{88}$$

It is understood that r and φ take on only a discrete set of values r_i, φ_i.

Second Approach

The second method will be applicable to evaluation of Eqs. (35)-(37). The basic step is the evaluation of $\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F}$. It will also be assumed that the density of possible scatterer locations is sufficient to enable the sums in the expression for $\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F}$ to be replaced approximately by integrals. Thus

$$\begin{aligned} &\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F} \\ &\approx \int P(r, \varphi)\Big\{ \cos\left[\omega_c t + r \sin(\omega_r t + \varphi)\right] \cos\left[\omega_c t' + r \sin(\omega_r t' + \varphi)\right] \\ &+ \sin\left[\omega_c t + r \sin(\omega_r t + \varphi)\right] \sin\left[\omega_c t' + r \sin(\omega_r t' + \varphi)\right]\Big\} r\, dr\, d\varphi \end{aligned} \tag{89}$$

where

$$\begin{aligned} A(r, \varphi, \mu, r', \varphi', \mu') &= 0, \quad (r, \varphi, \mu) \neq (r' \varphi' \mu') \\ &= P(r, \varphi)\, r\, d r\, d\varphi, \quad (r, \varphi, \mu) = (r', \varphi', \mu') \end{aligned} \tag{90}$$

It is understood that r dr dφ may actually represent a finite increment of area, equal in size to the area over which the a priori distribution of $\alpha(r, \varphi)$ is significantly correlated. The minimum size of r dr dφ would be about a half-wavelength square.

Equation (89) may be rewritten

$$\begin{aligned} \underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F} \approx \int P(r, \varphi) \cos\Big[\omega_c(t - t') &+ r \sin(\omega_r t + \varphi) \\ &- r \sin(\omega_r t' + \varphi)\Big] r\, dr\, d\varphi \end{aligned} \tag{91}$$

If it is assumed that

$$P(r, \varphi) = P(r), \qquad 0 \leqq \varphi \leqq 2\pi \tag{92}$$

then

$$\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F} \approx$$

$$2\pi \cos\left[\omega_c(t - t')\right] \int_0^\infty P(r)\, J_0 \left\{r\sqrt{2[1 - \cos \omega_r(t - t')]}\right\} r\, dr \tag{93}$$

$P(r, \varphi)$ reflects the a priori knowledge of the distribution of scatterer intensities. Some explicit expressions for $\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F}$ are:

(1) $P(r) = P_o\, r^n \exp\left[-\left(\frac{r}{r_o}\right)^2\right]$

$$\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F} = \pi r_o^{n+2} P_o \cos\left[\omega_c(t - t')\right] \Gamma\left(\frac{n}{2} + 1\right) \tag{94}$$

$$\times\ {}_1F_1 \left\{\frac{n + 2}{2}\ ;\ 1;\ -\frac{r_o^2}{2}\left[1 - \cos \omega_r(t - t')\right]\right\}$$

where ${}_1F_1$ is the confluent hypergeometric function. For $n = 0$,

$$\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F} \approx \pi\, r_o^2\, P_o \cos\left[\omega_c(t - t')\right] \exp\left[-\frac{r_o^2}{2}\right] \exp\left\{\frac{r_o^2}{2}\left[\cos \omega_r(t - t')\right]\right\} \tag{95}$$

(2) $P(r) = P_o \exp\left[-\left(\frac{r}{r_o}\right)\right]$

$$\underset{\sim}{F}^T \underset{\sim}{A} \underset{\sim}{F} = 2\pi\, r_o^2\, P_o \cos\left[\omega_c(t - t')\right] \left\{1 + 2r_o^2\left[1 - \cos \omega_r(t - t')\right]\right\}^{-3/2} \tag{96}$$

(3) $P(r) = P_o,\ 0 \leq r \leq r_o$ and $P(r) = 0,\ r > r_o$

$$\underset{\sim}{F}^T \underset{\sim}{A}\, \underset{\sim}{F} = 2\pi\, r_o\, P_o \cos\left[\omega_c (t - t')\right] J_1 \left\{ r_o \sqrt{2\left[1 - \cos \omega_r (t - t')\right]} \right\} \tag{97}$$

$$\times \left\{ 2 \left[1 - \cos \omega_r (t - t') \right] \right\}^{-\frac{1}{2}}$$

The result just stated for $P(r) = P_o \exp\left[- \left(\frac{r}{r_o}\right)^2 \right]$ can be further interpreted as follows.

Denoting $t - t'$ by τ,

$$\exp\left[\frac{r_o^2}{2} \cos \omega_r\, \tau \right] = \sum_{-\infty}^{\infty} I_n \left[\frac{r_o^2}{2} \right] \exp\left[i\, n\, \omega_r\, \tau \right] \tag{98}$$

Provided $n << \frac{r_o^2}{2}$,

$$I_n \left(\frac{r_o^2}{2} \right) \approx \left(\pi r_o^2 \right)^{-\frac{1}{2}} \exp\left(\frac{r_o^2}{2} \right) \tag{99}$$

Thus

$$\exp\left[\frac{r_o^2}{2} \cos \omega_r\, \tau \right] \approx \frac{1}{\omega_r} \left(\pi r_o^2 \right)^{-\frac{1}{2}} \exp\left(\frac{r_o^2}{2} \right) \delta^*(\tau) \tag{100}$$

where $\delta^*(\tau)$ represents a periodic Dirac delta function with period 2π. The approximation indicated in Eq. (100) is valid in the following sense. If both the δ^* function and the function on the left side of Eq. (100) are regarded as kernels of integral operators, the result of applying these two operators will be approximately the same, provided the function to which they are applied has a Fourier expansion (over the interval $2\pi/\omega_r$) which is negligible for all $n > n_o$ where $n_o << \frac{1}{2} r_o^2$.

Estimates $\hat{\alpha}(r_i, \varphi_i)$ involve application of this operator via Eq. (37) to $F(t, r_i, \varphi_i)$. This function in turn satisfies the criterion just stated only if $r_i << r_o$. Thus, use of expression Eq. (100) would be justified for estimates of $\underset{\sim}{\alpha}$ and functionals of $\underset{\sim}{\alpha}$ only for a region $r << r_o$.

C. COLLINEAR A PRIORI SCATTERER DISTRIBUTIONS WITH NO SPECULAR FLASH

The target model in the cases now to be considered is given by

$$S(t) = \sum_i a(r_i) \cos \left[\omega_c t + r_i \sin (\omega_r t + \theta_o) + \xi_i\right] \tag{101}$$

or equivalently

$$\begin{aligned} S(t) = &\sum_i \alpha(r_i, 1) \sin \left[\omega_c t + r_i \sin (\omega_r t + \theta_o)\right] \\ &+ \sum_i \alpha(r_i, 2) \cos \left[\omega_c t + r_i \sin (\omega_r t + \theta_o)\right] \end{aligned} \tag{102}$$

This is the same as in Eqs. (61) or (62), but with $\varphi_i \equiv 0$. Note that r_i can take on both positive and negative values (otherwise it would be necessary to admit two values of φ_i).

The phase factors ξ_i are still assumed to be random and independent for different indices i; this is what is meant by the statement that no specular flash is assumed, a priori, to occur. This is, however, an assumption about the state of a priori knowledge, and does not preclude cases in which a specular flash may occur in actual fact. A specular flash is, in this terminology, said to occur if there is some viewing aspect at which all the scatterers, or a dominant portion of them, are in phase.

The parameter θ_o is the target aspect at $t = 0$ and is considered to be an a priori unknown parameter. In the notation of Section II.B, $\lambda = \theta_o$. $\underset{\sim}{A}$ is still assumed to be given by Eq. (68).

The first approach of Section III.B can be followed with the result

$$\begin{aligned} B(r, 1, r', 1 \mid \theta_o) &= B(r, 2, r', 2 \mid \theta_o) \\ &= [A(r)]^{-1} + \frac{2}{N_o} I_{11}(r, r \mid \theta_o), \quad \text{if } r = r' \\ &= \frac{2}{N_o} I_{11}(r, r' \mid \theta_o) \quad \text{if } r \neq r' \end{aligned} \tag{103}$$

$$\begin{aligned} B(r, 1, r', 2 \mid \theta_o) &= -B(r, 2, r', 1 \mid \theta_o) \\ &= \frac{2}{N_o} I_{12}(r, r' \mid \theta_o) \end{aligned} \tag{104}$$

where I_{11} and I_{12} are given by Eqs. (80) and (81) with

$$\rho = | r - r' | \tag{105}$$

$$\theta^* = \theta_o \tag{106}$$

In particular, if $\theta_M = K\pi$, i.e., if the observation interval is an integral number of revolutions, then

$$I_{11}(r, r' \mid \theta_o) = \frac{K\pi}{\omega_r} J_o(| r - r' |) \tag{107}$$

$$I_{12}(r, r' \mid \theta_o) = 0 \tag{108}$$

Thus, in the case $\theta_M = K\pi$, $\underset{\sim}{B}$ does not depend on θ_o. Therefore Eqs. (29a) and (30a) of Section II.B are applicable in these cases. θ_M has the same meaning as in Section III.B.

Finally, when $\theta_M = K\pi$

$$\hat{\underset{\sim}{\alpha}}^T(\theta_o)\ \underset{\sim}{B}\ \hat{\underset{\sim}{\alpha}}(\theta_o) = \frac{4}{N_o^2} \sum_{i,j} \underset{\sim}{B}^{-1}(r_i, r_j)$$

$$\times \int\int S(t)\ S(t')\cos\Big[\omega_c(t - t') + r_i \sin(\omega_r t + \theta_o) - r_j \sin(\omega_r t' + \theta_o)\Big]\ dt\ dt' \tag{109}$$

This defines the a posteriori p.d.f. of θ_o given $\underset{\sim}{S}$ via Eq. (29).

D. COLLINEAR A PRIORI SCATTERER DISTRIBUTION WITH A SPECULAR FLASH

A specular flash would occur at some target aspect if, for example, $\xi_i = \xi$ for all i. However, if this assumption were made, the joint probability distribution of $\alpha(r_i, 1)$ and $\alpha(r_i, 2)$ would not be Gaussian (even though the marginal distributions for any one index i would be) unless also $a_i = a\ P(i)$, all i, where P(i) is an a priori known function. Here a_i has been written for $a(r_i)$.

The latter is not an altogether unrealistic requirement, since one would generally expect that the existence of an aspect where the scatterers are all in phase would not be accompanied by a very irregular variation of amplitudes of the individual scatterers. For example, the field from a uniformly illuminated antenna would be equivalent to a_i = constant.

This suggests a model in which the signal from a linear object giving a specular flash is represented by Eq. (101), but with

$$\xi_i = \xi, \quad \text{all } i \tag{110}$$

where ξ now represents the phase of a signal reflected from r = 0, and

$$a_i = a\, P(i, \underset{\sim}{\lambda}^*) \tag{111}$$

where $\underset{\sim}{\lambda}^*$ is a vector of unknown parameters having a small number of components, such as two or three.

In such a case, $\underset{\sim}{\alpha}$ is a two-component vector with a Gaussian a priori p.d.f.

$$\begin{aligned} \alpha_1 &= a \cos \xi \\ \alpha_2 &= a \sin \xi \end{aligned} \tag{112}$$

$\underset{\sim}{A}$ can be assumed to be a diagonal matrix, with

$$\begin{aligned} A_{11} &= A_{22} = \tfrac{1}{2} E(a^2) \\ A_{12} &= A_{21} = 0 \end{aligned} \tag{113}$$

The 2 × 2 matrix $\underset{\sim}{B}(\theta_o, \underset{\sim}{\lambda}^*)$, with components $B_{ij}(\theta_o, \underset{\sim}{\lambda}^*)$, is derived via the first approach in Section III.B. To illustrate, consider the case $\theta_M = K\pi$, where the observation is an integral number of periods. The result is then

$$\underset{\sim}{B}(\theta_o, \underset{\sim}{\lambda}^*) = B(\underset{\sim}{\lambda}^*) \tag{114}$$

$$B_{11}(\underset{\sim}{\lambda}^*) = B_{22}(\underset{\sim}{\lambda}^*)$$

$$= \frac{2}{E(a^2)} + \frac{K\pi}{\omega_r} \sum_{i,j} P(r_i, \underset{\sim}{\lambda}^*) \, P(r_j, \underset{\sim}{\lambda}^*) \, J_o(| \, r_i - r_j \, |) \tag{115}$$

$$B_{12}(\underset{\sim}{\lambda}^*) = B_{21}(\underset{\sim}{\lambda}^*) = 0 \tag{116}$$

Also, if $\underset{\sim}{\lambda} = (\theta_o, \underset{\sim}{\lambda}^*)$,

$$\hat{\alpha}_1(\underset{\sim}{\lambda}) = \hat{\alpha}_1(\theta_o, \underset{\sim}{\lambda}^*) \tag{117}$$

$$= \left(\frac{2}{N_o}\right) \left[B_{11}(\underset{\sim}{\lambda}^*)\right]^{-1} \int \sum_i P(r_i, \underset{\sim}{\lambda}^*) \sin \left[\omega_c \; t + r_i \sin \left(\omega_r \; t + \theta_o\right)\right] S(t) \, dt$$

$$\hat{\alpha}_2(\underset{\sim}{\lambda}) = \hat{\alpha}_2(\theta_o, \underset{\sim}{\lambda}^*) \tag{118}$$

$$= \left(\frac{2}{N_o}\right) \left[B_{22}(\underset{\sim}{\lambda}^*)\right]^{-1} \int \sum_i P(r_i, \underset{\sim}{\lambda}^*) \cos \left[\omega_c \; t + r_i \sin \left(\omega_r \; t + \theta_o\right)\right] S(t) \, dt$$

$$\hat{\underset{\sim}{\alpha}}^T(\underset{\sim}{\lambda}) \, \underset{\sim}{B}(\underset{\sim}{\lambda}) \, \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\lambda}) = \frac{4}{N_o^2} \left[B_{11}(\underset{\sim}{\lambda}^*)\right]^{-1} \int \int S(t) \, S(t') \left\{ \sum_{i,j} P(r_i, \underset{\sim}{\lambda}^*) \, P(r_j, \underset{\sim}{\lambda}^*) \right. \tag{119}$$

$$\left. \cos \left[\omega_c (t - t') + r_i \sin \left(\omega_r \; t + \theta_o\right) - r_i \sin \left(\omega_r \; t' + \theta_o\right)\right] \right\} dt \, dt'$$

It would be necessary to use Eq. (29) or Eq. (30) in this case, but the expression for det $\left[\underset{\sim}{B}(\underset{\sim}{\lambda})\right]$ is very simple:

$$\det \left[\underset{\sim}{B}(\underset{\sim}{\lambda})\right] = \left[B_{11}(\underset{\sim}{\lambda}^*)\right]^2 \tag{120}$$

Since the total number of unknown parameters in this case has been reduced, by assumption, to a rather small number--α_1, α_2, θ_o, and the components of $\underset{\sim}{\lambda}$--the use of the infoı ation matrix to compute lower bounds on estimation errors, and the use of maximum likelihood estimates, may be an adequate treatment.

It would also be possible to consider mixed cases, i.e., scattering bodies which, a priori, have a component of the type considered in this subsection, with a specular flash; and another component of the type treated in Section III.B. It would be possible to apply a mixture of the approaches described above. For example, for purposes of estimating $\underset{\sim}{\lambda}$ or θ_o for the component which provides the specular flash, it would be possible in some cases to consider the other component to provide the equivalent of noise (changing the noise inverse covariance matrix from $\underset{\sim}{\eta}$ to $\underset{\sim}{\eta}^*$, but integrating in Eq. (35) only over the scatterers in the second component). In certain cases, the resulting equivalent noise could still be considered white; see Eqs. (98)-(100).

E. PHASE ERRORS AND INCOHERENT PROCESSING

There are two possible sources of reduction in the extent of coherence maintained over the observation time: first, phase errors (such as those due to propagation medium inhomogeneities, or to phase jitter in the radar, or to uncorrected platform motion); and second, failure of the radar to attempt to preserve phase coherence over the observation time. An extreme case of the second type arises when the radar observes only the amplitude of the returned signal.

It is well known that considerable signature analysis is possible even with amplitude-only observations, and it is of interest to compare the results achieved with those achieved by fully coherent processing. The effect of phase errors on coherent processing is also of interest. The following discussion describes a method of approach to the analysis of these questions; however, many questions remain open, even with regard to the methodology.

Let us, for the sake of argument, revert to the signal model of Sections III.A and III.B, i.e., to the a priori noncollinear collection of scatterers. However, the signal (free of additive noise) will now be represented as

$$
\begin{aligned}
S(t) &= \sum_i a(r_i, \varphi_i) \cos \left[\omega_c t + r_i \sin (\omega_r t + \varphi_i) + \xi_i + \zeta(t)\right] \\
&= \sum_i \alpha(r_i, \varphi_i, 1) \sin \left[\omega_c t + r_i \sin (\omega_r t + \varphi_i) + \zeta(t)\right] \\
&\quad + \sum \alpha(r_i, \varphi_i, 2) \cos \left[\omega_c t + r_i \sin (\omega_r t + \varphi_i) + \zeta(t)\right]
\end{aligned}
\tag{121}
$$

where $\{\zeta(t)\}$ is an a priori unknown random process.

The distinction between the phase factors ξ_i and $\{\zeta(t)\}$ is as follows. The quantities ξ_i are fixed for all time and represent relative phases of the (rigidly connected) individual scatterers. The quantities $\zeta(t)$ are the same for all i, and represent phase deviations common to the whole scattering body, such as those that would be produced by motions along the line of sight of the center of rotation of the body. This formulation does not eliminate

the contribution of the phase terms $r_i \sin (\omega_r t + \varphi_i)$ to the information which can be extracted from signature analysis, even if the process $\{\zeta(t)\}$ is uniformly distributed and has very short correlation time. (If the phases were represented as processes $\{\zeta_i(t)\}$, with the processes independent for different values of i and having uniform distributions and short correlation times, then the phase terms $r_i \sin (\omega_r t + \varphi_i)$ would have no effect and the rotation of the body would be immaterial.)

Three questions of interest can be raised:

a. Suppose the radar is coherent (observes both amplitude and phase), and in fact forms Bayes estimates predicated on the assumption that $\{\zeta(t)\} \equiv 0$, but $\{\zeta(t)\}$ is in fact not zero. What estimation errors are produced by $\{\zeta(t)\}$?

The analytical procedure to be employed in answering this question is as follows. The estimates $\hat{\underset{\sim}{\alpha}}$ or $\hat{\beta}$ are the same as those given in Section II, specialized to rotating targets as in Section III.B, with $\{\zeta(t)\}$ assumed to be zero. Thus, estimates of linear forms in $\underset{\sim}{\alpha}$ are still linear forms in S(t), now interpreted to include additive noise $\{\varepsilon(t)\}$, and estimates of quadratic forms in $\underset{\sim}{\alpha}$ are still quadratic forms in S(t). In determining the moments of the estimation error, the same procedure is followed as in Section II.D, except that Eq. (50) can no longer be used. In place of this, one must use

$$\underset{\sim}{S} = \underset{\sim}{F} \{\underset{\sim}{\zeta}\} \underset{\sim}{\alpha}_o + \underset{\sim}{\varepsilon} \tag{50a}$$

where $\underset{\sim}{F} \{\underset{\sim}{\zeta}\}$ is formed by including the random phase process $\underset{\sim}{\zeta}$ in the factors multiplying each component of $\underset{\sim}{\alpha}$.

If $\zeta(t)$ is bounded (with high probability) to less than about one radian over the whole observation time, then $F(\zeta)$ can be expanded in a first order expansion

$$F(r_i, \varphi_i, \mu, \zeta, t) \approx$$

$$F(r_i, \varphi_i, \mu, 0, t) + \zeta(t)\, F_\zeta(r_i, \varphi_i, \mu, 0, t) \tag{122}$$

where F_ζ represents a derivative in an obvious way. The estimation errors are then linear or quadratic functions of both $\{\varepsilon(t)\}$ and $\{\zeta(t)\}$; their moments can be determined by the same techniques as in Section II.D. In such a case, it could not be said that coherent processing has been destroyed by $\{\zeta(t)\}$, but only that the results have been degraded by an additional noise term.

For coherence to be destroyed, it would be necessary to assume that $\{\zeta(t)\}$ is not bounded to a small value over the observational interval. If $\{\zeta(t)\}$ is not bounded in this manner, then the first-order expansion is not valid. However, it is still possible to get the first and second moments of linear or quadratic forms in $\underset{\sim}{S}$ in certain cases, e.g., if $\{\zeta(t)\}$ is a Gaussian process (defined over $-\infty$, ∞), or if $\zeta(t)$ has a correlation time which is short compared with the time in which $r_i \sin(\omega_r t + \varphi_i)$ changes significantly, for all i. Methods for doing this form another entire subject and will not be further described here.

Returning now to the other questions of interest:

b. Suppose the Bayes estimate is formed, utilizing the a priori representation Eq. (121) for the signal, with some definite a priori distribution attributed to $\{\zeta(t)\}$. What are the estimates then?

In principle, $\{\zeta(t)\}$ can be regarded as a parameter vector $\underset{\sim}{\lambda}$ of the type treated in Section II.B, but the problem still remains of elucidating the effect of this particular type of unknown parameter vector. This in turn is related to the third question:

c. Suppose the radar only observes amplitude?

The Bayes estimate formed by taking into account the presence of $\{\zeta(t)\}$ takes the following form. Let

$$\underset{\sim}{\Psi} = \underset{\sim}{F}^{T} \underset{\sim}{\eta} \underset{\sim}{F} = \underset{\sim}{B} - \underset{\sim}{A}^{-1} \tag{123}$$

Then, from Eq. (121), if ω_c is large

$$\begin{aligned} &\Psi(r_i, \varphi_i, 1, r_j, \varphi_j, 2) \\ &= \Psi(r_i, \varphi_i, 2, r_j, \varphi_j, 1) = 0 \end{aligned} \tag{124}$$

and

$$\begin{aligned} &\Psi(r_i, \varphi_i, 1, r_j, \varphi_j, 1) \\ &= \Psi(r_i, \varphi_i, 2, r_j, \varphi_j, 2) \\ &= \frac{1}{N_o} \int \cos \Big[r_i \sin (\omega_r t + \varphi_i) \\ &\quad - r_j \sin (\omega_r t + \varphi_j) \Big] dt \end{aligned} \tag{125}$$

This is exactly the same as if the process $\{\zeta(t)\}$ were identically zero. Moreover, it can be similarly verified that $\underset{\sim}{F}^{T} \underset{\sim}{A} \underset{\sim}{F}$ is independent of $\{\zeta(t)\}$, under the assumption that $\underset{\sim}{A}$ is diagonal. Thus, the matrices $\underset{\sim}{B}$ or $\underset{\sim}{\eta}^{*}$ entering into the expression for $\hat{\underset{\sim}{\alpha}}$ or $\hat{\beta}$ are independent of $\{\zeta(t)\}$; the only dependence of $\hat{\underset{\sim}{\alpha}}$ or $\hat{\beta}$ on $\{\zeta(t)\}$

arises from the dependence of F on $\{\zeta(t)\}$, in the expressions Eq. (9) or (37). To be specific, let

$$\underset{\sim}{\alpha} = (\underset{\sim}{\alpha}^{(1)}, \underset{\sim}{\alpha}^{(2)}) \tag{126}$$

where the components of $\underset{\sim}{\alpha}^{(1)}$ are $\alpha(r_i, \varphi_i, 1)$ and similarly for $\underset{\sim}{\alpha}^{(2)}$. Also, $\underset{\sim}{B}$ consists of two identical blocks corresponding to $\mu = 1$ and $\mu = 2$; let each of these blocks be denoted simply $\underset{\sim}{\widetilde{B}}$. Also, let

$$\begin{aligned} \widetilde{B}(r_i, \varphi_i, r_j, \varphi_j) &= \widetilde{B}_{ij} \\ \alpha^{(1)}(r_i, \varphi_i) &= \alpha_i^{(1)} \\ \alpha^{(2)}(r_i, \varphi_i) &= \alpha_i^{(2)} \end{aligned} \tag{127}$$

Then

$$\hat{\alpha}_i^{(1)}(\underset{\sim}{\zeta}) = \sum_j \left(\underset{\sim}{\widetilde{B}}^{-1}\right)_{ij} \cdot \frac{2}{N_o} \int \sin\left[\omega_c t + r_i \sin(\omega_r t + \varphi_i) + \zeta(t)\right] \cdot S(t)\, dt \tag{128}$$

$$\hat{\alpha}_i^{(2)}(\underset{\sim}{\zeta}) = \sum_j \left(\underset{\sim}{\widetilde{B}}^{-1}\right)_{ij} \cdot \frac{2}{N_o} \int \cos\left[\omega_c t + r_i \sin(\omega_r t + \varphi_i) + \zeta(t)\right] \cdot S(t)\, dt$$

Now further assume that the radar is a pulse radar, subject to the following conditions: (1) the pulses are sinusoids at frequency ω_c and have length much greater than the a priori range extent of the target; and (2) the pulse length τ_p is such that the radial motion of all a priori significant scatterer locations, and the change in $\zeta(t)$, are negligible compared to a wavelength during the time extent of a pulse. (Radial motion means motion along the line of sight to the radar.)

The matrix $\underset{\sim}{B}$ can be found from Eqs. (125) and (123); in this case, Eq. (125) takes the form

$$\Psi(r_i, \varphi_i, 1, r_j, \varphi_j, 1) = \Psi(r_i, \varphi_i, 2, r_j, \varphi_j, 2)$$

$$= \frac{\tau_p}{N_o} \sum_n \cos \left[r_i \sin (\omega_r t_n + \varphi_i) - r_j \sin (\omega_r t_n + \varphi_j) \right] \qquad (125a)$$

If the further assumption is made that the motion (projected on the line of sight) of all a priori significant scatterer locations is negligible compared to a wavelength during the interpulse period Δt, then Eq. (125a) can also be expressed

$$\Psi(r_i, \varphi_i, 1, r_j, \varphi_j, 1) = \Psi(r_i, \varphi_i, 2, r_j, \varphi_j, 2)$$

$$\approx \frac{\tau_p}{N_o \Delta t} \int \cos \left[r_i \sin (\omega_r t + \varphi_i) - r_j \sin (\omega_r t + \varphi_j) \right] dt \qquad (125b)$$

which is the same as Eq. (125) except for the constant factor $\tau_p/\Delta t$.

To see what form $\hat{\underset{\sim}{\alpha}}$ takes, let

$$S(t) = S_1(t) \sin \omega_c t + S_2(t) \cos \omega_c t \qquad (129)$$

Then, Eq. (128) becomes

$$\hat{\alpha}_i^{(1)}(\underset{\sim}{\zeta}) = \sum_j \left(\underset{\sim}{\widetilde{B}}^{-1}\right)_{ij} \cdot \frac{\tau_p}{N_o} \sum_n \left\{ S_{1n} \cos \left[r_i \sin (\omega_r t_n + \varphi_i) + \zeta_n \right] \right.$$

$$\left. + S_{2n} \sin \left[r_i \sin (\omega_r t_n + \varphi_i) + \zeta_n \right] \right\} \qquad (128a)$$

$$\hat{\alpha}_i^{(2)}(\underset{\sim}{\zeta}) = \sum_j \left(\underset{\sim}{\widetilde{B}}^{-1}\right)_{ij} \cdot \frac{\tau_p}{N_o} \sum_n \left\{ S_{2n} \cos \left[r_i \sin (\omega_r t_n + \varphi_i) + \zeta_n \right] \right.$$

$$\left. - S_{1n} \sin \left[r_i \sin (\omega_r t_n + \varphi_i) + \zeta_n \right] \right\}$$

where

$$S_{1n} = S_1(t_n)$$

$$S_{2n} = S_2(t_n)$$

$$\zeta_n = \zeta(t_n) \tag{130}$$

Next, consider estimates of $\left[\alpha_i^{(1)}\right]^2 + \left[\alpha_i^{(2)}\right]^2$. Let $\underset{\sim}{Q}^{(i)}$ be the quadratic form associated with this functional of $\underset{\sim}{\alpha}$. The term Trace $\left(\underset{\sim}{B}^{-1}\ \underset{\sim}{Q}^{(i)}\right)$ in Eq. (16) is independent of $\underset{\sim}{\zeta}$. The second term in Eq. (16) becomes

$$\begin{aligned}
&\hat{\underset{\sim}{\alpha}}^T(\underset{\sim}{\zeta})\ \underset{\sim}{Q}^{(i)}\ \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\zeta}) \\
&= \left(\frac{T_p}{N_o}\right)^2 \sum_{j,k} \sum_{m,n} \left(\tilde{\underset{\sim}{B}}^{-1}\right)_{ij} \left(\tilde{\underset{\sim}{B}}^{-1}\right)_{ik} \Big\{\left[S_{2m} \cos \rho_{jm} - S_{1m} \sin \rho_{jm}\right] \\
&\qquad\qquad \times \left[S_{2n} \cos \rho_{kn} - S_{1n} \sin \rho_{kn}\right]\Big\} \\
&+ \left(\frac{T_p}{N_o}\right)^2 \sum_{j,k} \sum_{m,n} \left(\tilde{\underset{\sim}{B}}^{-1}\right)_{ij} \left(\tilde{\underset{\sim}{B}}^{-1}\right)_{ik} \Big\{\left[S_{1m} \cos \rho_{jm} + S_{2m} \sin \rho_{jm}\right] \\
&\qquad\qquad \times \left[S_{1n} \cos \rho_{kn} + S_{2n} \sin \rho_{kn}\right]\Big\}
\end{aligned} \tag{131}$$

where

$$\rho_{in} = r_i \sin (\omega_r t_n + \varphi_i) + \zeta_n \tag{132}$$

This simplifies to

$$\begin{aligned}
&\hat{\underset{\sim}{\alpha}}^T(\underset{\sim}{\zeta})\, \underset{\sim}{Q}^{(i)}\, \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\zeta}) \\
&= \left(\frac{T_p}{N_o}\right)^2 \sum_{j,k} \sum_{m,n} \left(\underset{\sim}{\tilde{B}}^{-1}\right)_{ij} \left(\underset{\sim}{\tilde{B}}^{-1}\right)_{ik} \\
&\quad \left\{ \left[S_{1m} S_{1n} + S_{2m} S_{2n}\right] \cos\left[\rho_{jm} - \rho_{kn}\right] \right. \\
&\quad \left. + \left[S_{1m} S_{2n} + S_{2m} S_{1n}\right] \sin\left[\rho_{jm} - \rho_{kn}\right] \right\}
\end{aligned} \tag{133}$$

Similarly, the expression $\hat{\underset{\sim}{\alpha}}^T \underset{\sim}{B}\, \hat{\underset{\sim}{\alpha}}$ appearing in Eq. (29a), with $\underset{\sim}{\lambda} = \underset{\sim}{\zeta}$, is

$$\begin{aligned}
&\hat{\underset{\sim}{\alpha}}^T(\underset{\sim}{\zeta})\, \underset{\sim}{B}\, \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\zeta}) \\
&= \left(\frac{T_p}{N_o}\right)^2 \sum_{i,j} \sum_{m,n} \left(\underset{\sim}{\tilde{B}}^{-1}\right)_{ij} \left\{ \left[S_{1m} S_{1n} + S_{2m} S_{2n}\right] \cos\left[\rho_{im} - \rho_{jn}\right] \right. \\
&\qquad\qquad \left. + \left[S_{1m} S_{2n} + S_{2m} S_{1n}\right] \sin\left[\rho_{im} - \rho_{jn}\right] \right\}
\end{aligned} \tag{134}$$

Taking account of Eq. (132), Eqs. (133) and (134) can be written

$$\hat{\underset{\sim}{\alpha}}^T(\underset{\sim}{\zeta})\, \underset{\sim}{Q}^{(i)}\, \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\zeta}) = \sum_{m,n} \mathcal{B}^{(i)}_{mn} \cos\left(\zeta_m - \zeta_n + \beta^{(i)}_{mn}\right) \tag{133a}$$

$$\hat{\underset{\sim}{\alpha}}^T(\underset{\sim}{\zeta})\, \underset{\sim}{B}\, \hat{\underset{\sim}{\alpha}}(\underset{\sim}{\zeta}) = \sum_{m,n} \mathcal{D}_{mn} \cos\left(\zeta_m - \zeta_n + \gamma_{mn}\right) \tag{134a}$$

The a priori p.d.f. $q(\underset{\sim}{\lambda})$ in Eq. (29a), with $\underset{\sim}{\lambda} = \underset{\sim}{\zeta}$, can be assumed to be such that the components ζ_n of $\underset{\sim}{\zeta}$ are mutually independent and uniformly distributed over $(0, 2\pi)$. The evaluation of integrals of

the type appearing in Eq. (29a), resulting from substitution of Eqs. (133a) and (134a) into Eq. (29a), seems to be very difficult in general. The problem clearly amounts to evaluation of the expected values of $\cos \zeta_m \cos \zeta_n$, $\sin \zeta_m \sin \zeta_n$, and $\sin \zeta_m \cos \zeta_n$, with respect to the a posteriori distribution of $\{\zeta_n\}$ implicitly defined by inserting Eq. (134a) into Eq. (29a). However, there are some special cases in which such evaluation may be considerably simplified.

For example, suppose the a posteriori p.d.f. of $\underset{\sim}{\zeta}$ is also such that $\{\zeta_n\}$ are mutually independent and uniformly distributed. It is clear that this will not in general be true; however, it will tend to be true if the sampling interval Δt is such that a significant portion of a priori scatterer positions move along the line of sight by, say, a wavelength in time Δt.

Then, the unconditional estimate of $\left[\alpha_i^{(1)}\right]^2 + \left[\alpha_i^{(2)}\right]^2 = \beta_i$ would simply be, from Eq. (133),

$$\begin{aligned} \hat{\beta}_i &= \mathrm{Trace}\left[\underset{\sim}{B}^{-1}\, \underset{\sim}{Q}^{(i)}\right] \\ &+ \left(\frac{T_p}{N_o}\right)^2 \sum_n \sum_{j,k} \left(\underset{\sim}{\tilde{B}}^{-1}\right)_{ij} \left(\underset{\sim}{\tilde{B}}^{-1}\right)_{ik} \left[S_1^2(t_n) + S_2^2(t_n)\right] \\ &\times \cos\left[r_j \sin(\omega_r t_n + \varphi_j) - r_k \sin(\omega_r t_n + \varphi_k)\right] \end{aligned} \tag{135}$$

This depends only on the square amplitudes $S_1^2 + S_2^2$ of the observed signals.

Equation (135) represents the estimate achieved when $\beta_i(\underset{\sim}{\zeta})$ is integrated with respect to the a priori p.d.f. of $\{\zeta_n\}$. It is tempting to conjecture that this represents, in general, the optimum

Bayes estimate of β_i when incoherence is due to the fact that the radar observes only the signal amplitudes. When the components $\{\zeta_n\}$ are a posteriori, as well as a priori, independent and uniformly distributed, this conjecture must be true. The author has so far been unable to verify the truth of this conjecture in general, and in fact it is somewhat doubtful that it would prove to be true in general. It is easy to see that the Bayes estimate when the radar observes only amplitude must always be at least as good as the estimate given by Eq. (135).

If $\{\zeta_n\}$ are not a posteriori independent and uniformly distributed (as will be the case when the sampling rate is so high that a significant part of the a priori scatterer positions move along the line of sight by less than, say, one-quarter wavelength), then Eq. (135) does not represent the Bayes estimate if the radar observes both amplitude and phase. There are still cases in which it is probably justified to assume certain simplifications. For example, suppose the sampling rate is such that a significant part of the a priori scatterer positions move along the line of sight by the order of a wavelength in time $K\,\Delta t$, $K > 1$. Then, it is very probably true that ζ_m is statistically independent of ζ_n if $|m - n| \geq K$, and that the marginal a posteriori p.d.f. of any single ζ_n is uniform over $(0,2\pi)$. This would permit one to set equal to zero all terms for which $|m - n| \geq K$ in integrating Eq. (133) with respect to the a posteriori p.d.f. of $\underset{\sim}{\zeta}$.

IV. INITIAL STEPS TO IMPLEMENTATION OF COMPUTER PROGRAMS

It would be of interest to implement some of the techniques by programming them on a computer. Two approaches are indicated: (a) implementation of the methods for forming the estimates $\hat{\beta}$, and (b) implementation of the calculation of mean square error for linear and quadratic estimates.

The formation of the estimates themselves would be a Monte Carlo program since it would be necessary to produce simulated data with additive errors or phase errors (or both). Mean square error could, of course, be determined by repeated trials of such a Monte Carlo program. Calculation of mean square error via the formulas of Section II would not require Monte Carlo trials or generation of simulated data.

It would be most practical to attempt these implementations in an ascending order of complexity with respect to various features of the problem, such as:

<u>Type of A Priori Knowledge of Scatterer Distribution</u>

1. Noncollinear, radially symmetric distributions of scatterer locations.
2. Collinear distributions without a specular flash.
3. Collinear distributions with a specular flash.
4. Mixture of (1) and (3), or (2) and (3).
5. Surface of an opaque rotating sphere.

Observation Interval

1. Integral number of rotation periods, via Eqs. (35)-(37), (95), and (100).
2. Fractional period observation interval sufficiently short so that the ratio of observation time to the time required for all a priori significant scatterer locations to move along the line of sight by a half-wavelength is a small integer. The observation times could then be considered, with good approximation, to be a small number of instants, and Eqs. (35)-(37) could be used with direct matrix inversion to find $\underset{\sim}{\eta}^{*}$.
3. Integral number of rotation periods, with total number of significant a priori scattering points (separated by one-half wavelength) a small number. This could be done via Eq. (86) with direct matrix inversion to find $\underset{\sim}{B}^{-1}$.
4. Fractional observation period with noncollinear scatterer distribution, via Eqs. (86)-(88) to compute $\underset{\sim}{B}^{-1}$.

Degree of Coherence

1. Completely coherent.
2. Pulse radar with complete pulse-to-pulse incoherence due to phase errors in data (but with radar observation of received signal phase), via Eqs. (131)-(135) and (29).
3. Pulse radar with partial incoherence due to correlated phase errors in data.

These operations can be done for a variety of different quadratic functionals and for a variety of assumptions concerning the a priori state of knowledge and the true state of affairs. For example, it is

of interest to compare the accuracy with which a "true" collinear scatterer distribution, and the aspect angle θ_o of Eq. (101), can be estimated, both when the a priori distribution is noncollinear and when the a priori distribution is collinear.

It is of particular interest to attempt the evaluation of the integrals appearing in Eq. (29a), when Eqs. (133a) and (134a) are inserted. This will yield information not only on the comparison between incoherent and coherent processing to determine $\left[\hat{\alpha}_i^{(1)}\right]^2 + \left[\hat{\alpha}_i^{(2)}\right]^2$, but also on the accuracy with which individual phase error differences can be inferred, even if the phase errors are independent from pulse to pulse.

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10. ABSTRACT

A Bayes approach to parameter estimation, based on the minimum mean square error criterion, applied to the problem of identifying such radar target characteristics as dimensions, shape, rate of rotation, and axis of rotation, as well as the more usually considered range, range rate, and range acceleration. Emphasis is on estimation of linear or quadratic functionals of the scatterer amplitudes, such as the average scattering cross section attributable to specific regions of the target. Theoretically, results can be obtained for nonstationary cases, such as those involving specular flashes, for any signal-to-noise ratio, and for any degree of prior knowledge of target characteristics. However, most cases require solution of certain integral equations for which the computer programs are as yet unwritten. Initial steps toward computer implementation are discussed. Applications include satellite identification, radar astronomy, and radar ground mapping.

11. KEY WORDS

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